ALGEBRAIC GEOMETRY

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ABSTRACT. This is my personal note[1] of course *Advanced Geometry 3* taught by Prof. Fabio Perroni in spring 2024 at University of Trieste. And as the title, this course is focused on algebraic geometry, especially on algebraic varieties over $\mathbb{C}$. The latest version can be found on my website:

This note is not completely following what's on the blackbord. The materials might be re-organized or have some slightly changes. I will also add some supplementary materials which will be marked by *.

Attention: there are definitely a considerable number of mistakes in this note, and all due to me. If you have any comments, corrections or suggestions, please send them to zwei@sissa.it. Any feedback is appreciated!

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0. Preface

The main reference is [5].

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1. Affine Varieties

1.1. Algebraic subsets.

All ring will be assumed as commutative ring with unit.

Definition 1.1.1. A **closed algebraic subset** $X \subset \mathbb{C}^n$ is the set of zeroes of a finite numbers of polynomials

$$X = \{a = (a_1, \ldots, a_n) \mid f_i(a_1, \ldots, a_n) = 0, \forall i = 1, \ldots, m\}$$

where $f_i \in \mathbb{C}[x_1, \ldots, x_n]$.

It is also denoted by $V(f_1, \ldots, f_m)$.

Remark 1.1. The ideal generated by $f_1, \ldots, f_m$ is

$$I = (f_1, \ldots, f_m) = \{\sum g_i f_i \mid g_i \in \mathbb{C}[x_1, \ldots, x_n]\}.$$

And the set of zeroes of I is $X = V(I) = \{a \in \mathbb{C}^n \mid f(a) = 0, \forall f \in I\}$.

By Hilbert basis theorem, every ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$ is f.g., i.e., $\exists f_1, \ldots, f_m$ s.t. $I = (f_1, \ldots, f_m)$. Hence we will talk about $V(I), I \subset \mathbb{C}[x_1, \ldots, x_n]$.

Proposition 1.1.1. *Let $I_1, I_2, \{I_\alpha\}_{\alpha \in A}$ be ideas of $\mathbb{C}[x_1, \ldots, x_n]$. $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$. Then the following hold true.*

(1) *If $I_1 \subset I_2$, then $V(I_2) \subset V(I_1)$,*
(2) $V(I_1) \cup V(I_2) = V(I_1 \cap I_2) = V(I_1 \cdot I_2)$ $(I_1 I_2 = \{fg | f \in I_1, g \in I_2\})$,
(3) $V(\sum_{\alpha \in A} I_\alpha) = \bigcap_{\alpha \in A} V(I_\alpha)$,
(4) *If $\mathfrak{m}_a := (x_1 - a_1, \ldots, x_n - a_n)$, then $V(\mathfrak{m}_a) = \{a\}$,*
(5) $V(\sqrt{I}) = V(I)$
$(\sqrt{I} = \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid f^K \in I \text{ for some } K > 0\})$.

Proof. (1) evident.
(2) Since $I_1 I_2 \subset I_1 \cap I_2 \subset I_1, I_2$, (1) implies that

$$V(I_1 I_2) \supset V(I_1 \cap I_2) \supset V(I_1), V(I_2).$$

Conversely, let $a \in V(I_1 I_2)$. If $a \notin V(I_1 \cap I_2)$, then $\exists f \in I_1 \cap I_2$ s.t. $f(1) \neq 0$. Then $f^2(a) \neq 0$, but $f^2 \in I_1 I_2$. The remain is similar.
(3) $(\subset)$ $I_\alpha \subset \sum I_\alpha \forall \alpha$, hence $V(\sum I_\alpha) \subset V(I_\alpha) \forall \alpha$.
$(\supset)$ Immediately.
(4) $b \in V_{\mathfrak{m}_a}$ iff $b_i - a_i = 0,\ \forall i$.
(5) $(\subset)$ $\sqrt{I} \supset I$
$(\supset)$ Let $a \in V(I)$. If $a \notin V(\sqrt{I})$, then $\exists f \in \sqrt{I}$ s.t. $f(a) \neq 0$. Hence $f^K(a) \neq 0$. contradiction. □

Remark 1.2. (1) It can happen that $I_1 I_2 \subsetneqq I_1 \cap I_2$,
(2) $\sqrt{I}$ is an ideal and it is called the radical of I,

(3) **Proposition** 1.1.1(2), (3) implies that algebraic subsets of $\mathbb{C}^n$ satisfy the axiom of closed sets of a topology on $\mathbb{C}^n$ and it is called **Zariski topology**.

Remark 1.3. The Zariski topology is not Hausdorff unless the base field $\mathbb{k}$ is finite.

1.2. Affine varieties.

Definition 1.2.1. An **affine variety** is a non-empty closed algebraic set $X \subset \mathbb{C}^n$ of the form $X = V(P)$ with P prime ideal.

Example 1.2.1. *Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be irrd. Then $V(f) = V((f)) \subset \mathbb{C}^n$ is an affine variety and it is called an hypersurface of $\mathbb{C}^n$.*

Note that if f is not irrd then (f) is not prime.

Example 1.2.2. *Let $g_2, \ldots, g_n \in \mathbb{C}[x_1]$. Consider $X := \{(a, g_2(a), \ldots, g_n(a)) \in \mathbb{C}^n \mid a \in \mathbb{C}\}$.*

It is a closed algebraic subset by $X = V(x_2 - g_2(x_1), \ldots, x_n - g_n(x_1))$. And since $\mathbb{C}[x_1, \cdots x_n]/(x_2 - g_2(x_1), \ldots, x_n - g_n(x_1)) \cong \mathbb{C}[x_1]$ which is a integral domain. Hence X is an affine variety and it is called rational space curve.

Exercise 1.2.1. *Let $\varphi_1, \ldots, \varphi_k \subset \mathbb{C}[x_1, \ldots, x_n]$ be homogeneous polynomials of degree 1. Suppose that $\{\varphi_i\}$ are linearly independent as elements of $(\mathbb{C}^n)^*$. Then for any $b_1, \ldots, b_k \in \mathbb{C}$, fixed $X = V(\varphi_1 - b_1, \ldots, \varphi_k - b_k)$, which is the set of solutions of the linear system $\varphi_i = b_i$.*

Prove that X is an affine variety. It is called a linear subspace of $\mathbb{C}^n$ of dimension $n - k$.

Now for any subset $S \subset \mathbb{C}^n$, we can define

$$I(S) := \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid f(a) = 0, \forall a \in S\}.$$

We have the following amazing theorem.

Theorem 1.2.1 (Hilbert's Nullstellensatz)**.** *For any ideal $J \subset \mathbb{C}[x_1, \ldots, x_n]$,*

$$I(V(J)) = \sqrt{J}.$$

In particular, if the ideal J is prime, then $I(V(J)) = J$.

Remark 1.4. (1) The theorem holds true for any algebraic closed field(See [1]).

(2) It fails if the field is not algebraic closed. For example, take $\mathbb{k} = \mathbb{R}$, $I(V(x^2 + y^2 + 1)) = \mathbb{R}[x, y]$ where $V(x^2 + y^2 + 1)$ is actually empty.

(3) (Study's lemma) Let $\mathbb{k} = \bar{\mathbb{k}}$. If $f \in \mathbb{k}[x_1, \ldots, x_n]$ is irrd, then $I(V(f)) = (f)$

Lemma* 1.2.1. *If $Y_1 \subset Y_2$ are algebraic subsets of $\mathbb{C}^n$, then $I(Y_1) \supset I(Y_2)$.*

Proposition* 1.2.1. $V(I(S)) = \bar{S}$

Proof. On the one hand we have $S \subset V(I(S))$ where by definition S is closed. Hence $\bar{S} \subset V(I(S))$. On the other hand, recall that the closure

$$\bar{S} = \bigcap W$$

where W runs over all algebraic subsets of $\mathbb{C}^n$ that contain S. And we can write $W = V(J)$ for some ideal J. Then $S \subset V(J)$ and by **Lemma* 1.2.1**, we have $I(S) \supset I(V(J)) \supset J$. Then by **Proposition 1.1.1**(1), $W = V(J) \subset V(I(S))$ for any such W. It follows the statement. □

Definition 1.2.2. Let $V(P) \subset \mathbb{C}^n$ be an affine variety. And let $\Bbbk \subset \mathbb{C}$ be a subfield. A point $a \in V(P)$ is called a **$\Bbbk$-generic point** if the following condition holds true: $\forall f \in \Bbbk[x_1, \dots, x_n]$, if $f(a) = 0$, then $f \in P$.

Example 1.2.3. *Consider $g_2, \dots, g_n \in \mathbb{Q}[x_1]$ and let $X = V(x_2 - g_2(x_1), \dots, x_n - g_n(x_1))$ be the rational space curve. Let $a := (\pi, g_2(pi), \dots, g_n(\pi)) \in X$. Then a is $\mathbb{Q}$-generic.*

Indeed, let $f \in \mathbb{Q}[x_1, \dots, x_n]$ is s.t. $f(a) = 0$. But $\varphi := f(x_1, x_2 - g_2(x_1), \dots, x_n - g_n(x_1)) \in \mathbb{Q}[x_1]$, hence $\varphi = 0$. It follows that $\varphi \in P$.

Proposition 1.2.1. *Let $V(P)$ be an affine variety. Let $\Bbbk \subset \mathbb{C}$ be a subfield s.t. $\operatorname{tr.deg} \mathbb{C}|\Bbbk = \infty$. Then there exists $a \in V(P)$ a $\Bbbk$-generic point.*

Proof. Let $P = (f_1, \dots, f_m)$ and, WLOG, assume that $f_1, \dots, f_m \in \Bbbk[x_1, \dots, x_n]$

(Otherwise let $\Bbbk'$ be the minimal subfield of $\mathbb{C}$ containing $\Bbbk$ and the coefficients of $f_1, \dots, f_m$. Then $\operatorname{tr.deg} \mathbb{C}|\Bbbk' = \infty$ and any $\Bbbk'$-generic point is also a $\Bbbk$-generic point).

Let $P_0 = P \cap \Bbbk[x_1, \dots, x_n]$, which is prime. And let K be the fraction field of $\Bbbk[x_1, \dots, x_n]/P_0$.

Since for any $f/g \in K$, it is a root of $gy - f \in \Bbbk(\bar{x}_1, \dots, \bar{x}_n)[y]$, where $\bar{x}_1, \dots, \bar{x}_n$ is the isomorphic class in $\Bbbk[x_1, \dots, x_n]/P_0$. We have that $K|\Bbbk(\bar{x}_1, \dots, \bar{x}_n)$ is algebraic. Hence $\operatorname{tr.deg} K|\Bbbk \le n < \infty$.

In this situation, there exists a field homomorphism

$$\phi : K \to \mathbb{C}$$

s.t. $\phi|_K = \mathrm{id}_K$(Indeed, let $\lambda_1, \dots, \lambda_\delta \in K$ be a transcendence basis for $K|\Bbbk$. Let $z_1, \dots, z_\delta \in \mathbb{C}$ be algebraically independent over $\Bbbk$. The map $\lambda_i \mapsto z_i$, $\forall i$ extends to a unique field homomorphism from $K \to \mathbb{C}$. See [6] Ch.2 Thm 33).

Let $a_i := \phi(\bar{x}_i) \in \mathbb{C}$.

Claim. $a = (a_1, \dots, a_n) \in X$ is a $\Bbbk$-generic point.

Indeed. First we have that $f_i(\bar{x}_1, \dots, \bar{x}_n) = 0$ $i = 1, \dots, m$ in $\Bbbk[x_1, \dots, x_n]/P_0$. It follows that

$$0 = \phi(f_i(\bar{x}_1, \dots, \bar{x}_n)) = f_i(\phi(\bar{x}_1), \dots, \phi(\bar{x}_n)) = f_i(a_1, \dots, a_n) \ i = 1, \dots, m.$$

Hence $a \in X$.

Now let $f \in \Bbbk[x_1, \dots, x_n]$ s.t. $f(a) = 0$. If $f \notin P_0$, then $[f] \in \Bbbk[x_1, \dots, x_n]$ is nonzero. Applying ϕ to this class we get that $f(a) = 0$, which is contradiction. □

Remark 1.5. One could have defined $\Bbbk$-generic point for all $V(I)$ where $I \subset \mathbb{C}[x_1, \dots, x_n]$ is any ideal. But in the following case, it doesn't exist.

Let $I = (xy) \subset \mathbb{C}[x, y]$ be an ideal and $a = (a_1, a_2)$. If $a_2 = 0$, then for $y \subset \Bbbk[x, y]$, $\forall \Bbbk \subset \mathbb{C}$, $y(a) = a_2 = 0$, but $y \notin I$. It is similar when $a_1 = 0$.

Now we can give a proof of **Theorem** 1.2.1.

Proof. **Step 1.** Let $J = P$ be prime. Let $f \in I(V(P))$ and $\Bbbk$ be the minimal subfield of $\mathbb{C}$ containing $\mathbb{Q}$ and the coefficients of f. Then $\operatorname{tr.deg} \mathbb{C}/\Bbbk = \infty$ and by **Proposition** 1.2.1, there exists a $\Bbbk$-generic point $a \in X$. And since $f \in I(X)$, $f(a) = 0$, then $f \in P$.

Step 2. Not let J be any ideal and $f \in I(V(J))$. Consider the primary rep

$$\sqrt{J} = P_1 \cap \cdots \cap P_N.$$

Then $V(J) = V(\sqrt{J}) = V(P_1) \cup \cdots \cup V(P_N)$. So $f \in I(V(P_i))$ $i = 1, \dots, N$. Then by **Step 1.**, $f \in P_i$ $i = 1, \dots, N$, and $f \in \sqrt{I}$. □

Corollary 1.2.1. *There is an order-reversing correspondence*

$$\{J \subset \mathbb{C}[x_1, \dots, x_n] \mid J = \sqrt{J}\} \leftrightarrow \{\text{closed algebraic subset of } \mathbb{C}^n\}$$
$$J \mapsto V(J)$$
$$I(X) \leftarrow\!\shortmid X$$

Definition 1.2.3. Let $X = V(P) \subset \mathbb{C}^n$ be an affine variety with $P \subset \mathbb{C}[x_1, \dots, x_n]$ prime ideal. The ring $R_X := \mathbb{C}[x_1, \dots, x_n]/P$ is the **affine coordinate ring** of X.

Corollary 1.2.2. *In this situation, R_X is isomorphic to the ring of functions $X \to \mathbb{C}$ which are restrictions of polynomials in $\mathbb{C}[x_1, \dots, x_n]$.*

Proof. Let $\mathcal{F}(X) := \{F : X \to \mathbb{C} \mid$ s.t. $\exists f \in \mathbb{C}[x_1, \dots, x_n]$ s.t. $F(a) = f(a), \forall a$.

Restriction yields an surjective homomorphism

$$\mathbb{C}[x_1, \dots, x_n] \to \mathcal{F}(X) \to 0$$

and its kernel is P. Then we have the isomorphism. □

1.3. **Tangent spaces of affine varieties.**

Definition 1.3.1. Let $X = V(P)$ be an affine variety with $P \in \mathbb{C}[x_1, \dots, x_n$ prime. Let $a \in X$, the **Zariski tangent space** of X at a is the linear subspace of $\mathbb{C}^n$ given by the equations

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i) = 0, , \ \forall f \in P$$

and denoted by $T_{X,a}$[1].

[1] I prefer $T_a X$ so I might change this symbol hereafter

Remark 1.6. (1) If $P = (f_1, \dots, f_m)$, then

$$T_a X = V(\{\sum_{i=0}^{m} \frac{\partial f_j}{\partial x_i}(a)(x_i - a_i) = 0 \mid j = 1, \dots, m\}).$$

Indeed, ($\subset$) is obvious. ($\supset$) If $(b_1, \dots, b_n) \in \mathbb{C}^n$ is s.t.

$$\sum_{i=0}^{n} \frac{\partial f_j}{\partial x_i}(a)(b_i - a_i) = 0, \ \forall j = 1, \dots, m.$$

Let $f \in P$, we can write $f = \sum_{i=1}^{m} f_i g_i$ for some $g_i \in \mathbb{C}[x_1, \dots, x_n]$. Then

$$\sum_{i=0}^{n} \frac{\partial f}{\partial x_i}(a)(b_i - a_i) = \sum_{i=0}^{n}\sum_{i=1}^{m} \frac{\partial f_i g_i}{\partial x_i}(a)(b_i - a_i) = 0.$$

(2) $T_a X \subset \mathbb{C}^n$ is an affine subspace passing through a.

1.4. **Tangent spaces and derivations.** Let $R := R_X$ be the affine coordinate ring of X.

Recall that a **derivation** of R (centered) at $a \in X$ is a $\mathbb{C}$-linear map

$$D : R \to \mathbb{C}$$

s.t.

(1) $D(fg) = f(a)D(g) + g(a)D(f), \ \forall f, g \in R,$
(2) $D(\lambda) = 0, \ \forall \lambda \in \mathbb{C}.$

Let $Der_{R,a}$ be the set of such derivations.

Remark 1.7. $Der_{R,a}$ is a vector space over $\mathbb{C}$.

Proposition 1.4.1. *Let $\bar{x}_1, \dots, \bar{x}_n \in R$ be the classes of $x_1, \dots, x_n$. Then the map*

$$\varphi : Der_{R,a} \to \mathbb{C}^n$$
$$D \mapsto (D(\bar{x}_1), \dots, D(\bar{x}_n))$$

is an injective linear map and its image is $T_a X - a$.

Proof. Exercise. $\square$

1.5. **Dimension theory.** The Zariski tangent space we have defined before is an affine subspace of $\mathbb{C}^n$. As a vector space, it has dimension

$$\dim T_a X = n - \mathrm{rk}(\frac{\partial f_j}{\partial x_i(a)})_{i,j}.$$

For any $k \in \mathbb{N}$, we have

$$\begin{aligned}\{a \in X \mid \dim T_a X \geq k\} &= \{a \in X \mid \mathrm{rk}(\frac{\partial f_j}{\partial x_i(a)})_{i,j} \leq n - k\} \\ &= \{a \in X \mid \text{the determinants of all minors of ?} \\ &\quad (n-k+1) \times (n-k+1) \text{ of } \frac{\partial f_j}{\partial x_i(a)} \text{ are } 0\}.\end{aligned}$$

Hence $\{a \in X \mid \dim T_a X \geq k\}$ is a closed subset of X in the Zariski topology of X.

Remark 1.8. (1) $\{a \in X \mid \dim T_a X \geq k\} \subset \{a \in X \mid \dim T_a X \geq k-1\}$,
(2) Let $d := \min\{\dim T_a X \mid a \in X\}$. Observe that

$$U := \{a \in X \mid \dim T_a X = d\} = X - \{a \in X \mid \dim T_a X \geq d+1\}$$

is open and nonempty.

Proposition 1.5.1. *Let $X = V(P)$ be an affine variety with $P \subset \mathbb{C}[x_1, \ldots, x_n]$ prime. Let $\mathbb{C}(X) = \mathrm{Frac}(R_X)$. ($\mathbb{C}(X)$ is called the field of rational functions of X) Then*

$$d = \mathrm{tr.deg}(\mathbb{C}(X)/\mathbb{C}).$$

Definition 1.5.1. The dimension of an affine variety X is $\dim X := \mathrm{tr.deg}(\mathbb{C}(X)|X)$.

And a point $a \in X$ is smooth if $\dim T_a X = \dim X$. $a \in X$ is singular if $\dim T_a X > \dim X$.

Remark 1.9. Let $\bar{x}_1, \ldots, \bar{x}_n \in R_X$ be the classes of $x_1, \ldots, x_n$. Then $\mathbb{C}(X) = \mathbb{C}[\bar{x}_1, \ldots, \bar{x}_n]$.

Indeed. ($\subset$) is clear.

($\supset$) Let $\frac{\bar{f}}{\bar{g}} \in \mathbb{C}(X)$ where $\bar{f}, \bar{g} \in R_X$ and $\bar{g} \neq 0$. And $\bar{f}, \bar{g}$ are the classes of f, g respectively. Then $\bar{f}.\bar{g}$ are polynomials in $\bar{x}_1, \ldots, \bar{x}_n$. Then $\frac{\bar{f}}{\bar{g}} \in$ $bC[\bar{x}_1, \ldots, \bar{x}_n]$.

It implies that $\mathrm{tr.deg}(\mathbb{C}(X)|\mathbb{C}) < \infty$.

Example 1.5.1. (1) $\dim \mathbb{C}^n = n$
(2) $\forall a \in \mathbb{C}^n$, $\dim\{a\} = 0$ *(Jacobian is the identity)*
(3) *Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be irrd($f \notin \mathbb{C}$). Let $X = V(f)$.*

$$0 \leq \mathrm{rk}(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_1}) \leq 1$$

Notice that there exists $a \in X$ s.t. $\mathrm{rk}(\frac{\partial f}{\partial x_i}) = 1$.

Indeed. If $\mathrm{rk}(\frac{\partial f}{\partial x_i}) = 0, \forall a \in X$, then $\frac{\partial f}{\partial x_i} \in I(X) = (f)$. Hence $f | \frac{\partial f}{\partial x_i}, \forall i$. It follows that $\frac{\partial f}{\partial x_i} = 0, \forall i$ sicne $\deg \frac{\partial f}{\partial x_i} < \deg f$ if $\frac{\partial f}{\partial x_i} \neq 0$. Then $f \in \mathbb{C}$ contradiction.

Therefore, $\dim X = n - 1$.

(4) *Consider the rational space curve $X = V(x_2 - g_2(x_1, \ldots, x_n - g_n(x_1))$. Its Jacobian is*

$$\begin{pmatrix} -\frac{\partial g_2}{\partial x_1} & 1 & 0 & \cdots & 0 \\ -\frac{\partial g_2}{\partial x_1} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial g_n}{\partial x_1} & 0 & 0 & \cdots & 1 \end{pmatrix}$$

of rank $n-1$. Hence $\dim X = n - 1$.

(5) *Consider the cuspidal cubic curve* $X = V(x^2 - y^3) \subset \mathbb{C}^2$. *Its Zariski tangent space at* $p = (a^3, a^2)$ *is*

$$T_pX = \{(x,y) \in \mathbb{C}^2 \mid 2a^3(x-a^3) - 3a^4(y-a^2) = 0\}$$
$$= \begin{cases} \mathbb{C}^2, & a = 0, \\ 2a^3(x-a^3) - 3a^4(y-a^2) = 0, & a \neq 0. \end{cases}$$

Then $\min\{\dim T_pX\} = 1$ *and* $\dim X = n-1$. *X is singular at* $(0,0)$

Lemma 1.5.1. *Let R be an integral domain over field* $\Bbbk$ *and* $P \subset R$ *a prime ideal. Let* $K := \mathrm{Frac}(R)$ *and* $K' = \mathrm{Frac}(R/P)$. *Assume* $\mathrm{tr.deg}\, K|\Bbbk < \infty$. *Then*

$$\mathrm{tr.deg}\, K|\Bbbk \geq \mathrm{tr.deg}\, K'|\Bbbk$$

and the equality holds iff $P = (0)$.

Proof. If $P = (0)$ everything is clear. Assume $P \neq (0)$ and assume by contradiction that

$$\mathrm{tr.deg}\, K|\Bbbk < \mathrm{tr.deg}\, K'|\Bbbk$$

By Ch.II, Sec 12, Thm 27 of [6], there exist $\varphi_1, \ldots, \varphi_n \in R/P$ that are algebraically independent over $\Bbbk$ where $n = \mathrm{tr.deg}\, K|\Bbbk$. Let $f_1, \ldots, f_n \in R$ s.t. their classes in R/P are $\varphi_1, \ldots, \varphi_n$ respectively. Let $p \in P$, $p \neq 0$. Then $p, f_1, \ldots, f_n$ are algebraically dependent. Hence there exists a polynomial $\Phi \in \Bbbk[y, x_1, \ldots, x_n] \setminus 0$ s.t. $\Phi(p, f_1, \ldots, f_n) = 0$. WLOG, we can assume Φ is irrd (since R is an integral domain). Moreover $\Phi \neq \alpha y$, $\alpha \in \Bbbk$ since $p \neq 0$. Hence $\Phi(0, x_1, \ldots, x_n) \neq 0$. And passing to R/P, $\Phi(0, \varphi_1, \ldots, \varphi)n) = 0$, contradiction. □

Proposition 1.5.2. *Let* X, Y *be two affine varieties with* $X \subsetneqq Y$. *Then* $\dim X < \dim Y$.

Proof. Let $X = V(P)$, $Y = V(Q)$ with $P, Q \subset \mathbb{C}[x_1, \ldots, x_n]$ prime. Then $Q \subsetneqq P$. We have

$$0 \to \bar{P} \to R_Y \to R_X \to 0$$

where $\bar{P} = P/Q$. Then $R_X = R_Y/\bar{P}$.

By **Lemma** 1.5.1, $\mathrm{tr.deg}(\mathbb{C}(Y)|\mathbb{C}) \geq \mathrm{tr.deg}(\mathbb{C}(X))|\mathbb{C}$ and the equality holds iff $\bar{P} = (0)$, which is $P = Q$. □

Corollary 1.5.1. *Let* $X \subset \mathbb{C}^n$ *be an affine variety of dimension* $n-1$. *Then X is a hypersurface(i.e.* $\exists f \in \mathbb{C}[x_1, \ldots, x_n]$ *irrd s.t.* $X = V(f)$ *).*

Proof. Let $X = V(P)$ with P prime. Let $f \in P$, $f \neq 0$. Then $X \subset V(f)$. And there exist $f_1, \ldots, f_N \in \mathbb{C}[x_1, \ldots, x_n]$ irrd s.t.

$$f = f_1 \cdots f_N \in P.$$

Since P is prime, there exists $i \in \{1, \ldots, N\}$ s.t. $f_i \in P$. Hence $X \subset V(f_i)$. And since $\dim X = n - 1 = \dim V(f_i)$, by **Proposition** 1.5.2, we have $X = V(f_i)$. □

Corollary 1.5.2. *Let* $X \subset \mathbb{C}^n$ *be an affine variety. Then* $\dim X = 0 \Leftrightarrow X = \{a\}$ *for some* $a \in \mathbb{C}^n$

Proof. ($\Leftarrow$) Clear. ($\Rightarrow$) If $\exists a \in X$ and $\{a\} \neq X$, then $0 = \dim\{a\} < \dim X = 0$, contradiction. $\square$

Remark 1.10. Let $X = V(P) \subset \mathbb{C}^n$ be an affine variety with P prime. And $\dim X = n - r$. In general, there are no $f_1, \dots, f_r \in \mathbb{C}[x_1, \dots, x_n]$ s.t. $P = (f_1, \dots, f_r)$.

For example, Let $X \subset \mathbb{C}^3$ be an affine variety with $\dim X = 1$. If $P = I(X)$, the minimal number of generators of P is 3. Consider the map

$$\begin{aligned} \varphi : \mathbb{C} &\to \mathbb{C}^3 \\ a &\mapsto (a^3, a^4, a^5) \end{aligned}$$

Let $X := \{(a^3, a^4, a^5) \mid a \in \mathbb{C}\} \subset \mathbb{C}^3$. Then clearly we have that $X \subset V(I)$ where $I = (xz - y^2, x^3 - yz, x^2y - z^2)$. Conversely, let $(x, y, z) \in V(I)$, set $a := \frac{y}{x}$ if $x \neq 0$ (if $x = 0$ then $y = z = 0$). Then we have

$$\begin{aligned} a^3 &= \frac{y^3}{x^3} = \frac{xzy}{x^3} = x, \\ a^4 &= xa = y, \\ a^3 &= ya = z. \end{aligned}$$

Therefore, $X = V(I)$.

Moreover, I is a prime ideal and it cannot be generated by 2 polynomials.

Claim. $\sqrt{I}$ is prime.

Indeed. By **Theorem** 1.2.1, $\sqrt{I} = I(X)$. If $\exists f_1, f_2 \in \mathbb{C}[x, y, z]$ s.t. $f_1 f_2 \in \sqrt{I}$ but $f_1, f_2 \notin \sqrt{I}$. Then $f_1 \circ \varphi, f_2 \circ \varphi \in \mathbb{C}[t] \setminus 0$ but $(f_1 f_2) \circ \varphi = (f_1 \circ \varphi)(f_2 \circ \varphi) = 0$, contradiction. $\square$

Claim. $\sqrt{I}$ cannot be generated by 2 polynomials.

Indeed. Let $f \in \sqrt{I}$. It can be written as

$$f = \sum c_{ijk} x^i y^j z^k$$

s.t. $\sum c_{ijk} t^{3i+4j+5k} = 0$, $\forall t$. i.e., $\forall m \geq 0$, $\forall (i, j, k)$ s.t. $3i + 4j + 5k = m$,

$$\sum_{\substack{(i,j,k) \\ 3i+4j+5k=m}} c_{ijk} = 0, \ \forall m \geq 0.$$

(1) $m = 0$. $c_{000} = 0$.
(2) $m = 1, 2$. None.
(3) $m = 3$. $c_{100} = 0$.
(4) $m = 4$. $c_{010} = 0$.
(5) $m = 5$. $c_{001} = 0$.
(6) $m = 6$. $c_{200} = 0$.
(7) $m = 7$. $c_{110} = 0$.
(8) $m = 8$. $c_{101} + c_{020} = 0$. We get $\mathbb{C}(xz - y^2)$.

(9) $m = 9$. $c_{300} + c_{011} = 0$. We get $\mathbb{C}(x^3 - yz)$.
(10) $m = 10$. $c_{210} + c_{002} = 0$. We get $\mathbb{C}(x^2y - z^2)$.

In conclusion, f has the form

$$f = \alpha(xz - y^2) + \beta(x^3 - yz) + \gamma(x^2y - z^2) + \tilde{f},\ \alpha, \beta, \gamma \in \mathbb{C}.$$

If $\sqrt{I} = (f, g)$, then

$$g = \alpha'(xz - y^2) + \beta'(x^3 - yz) + \gamma'(x^2y - z^2) + \tilde{g}.$$

and we can express $xz - y^2, x^3 - yz, x^2y - z^2$ as a linear cpmbination of f, g. But they are linearly indenpendent. Contradiction. □

To prove **Proposition** 1.5.1, we need the following lemmas.

Lemma 1.5.2. *Let $U_1, U_2 \subset X$ be nonempty Zariski open subsets. Then $U_1 \cap U_2 \neq \varnothing$.*

Proof. Let $X = V(P)$ with P prime. We can write the open sets as

$$U_i = X \cap (\mathbb{C}^n \setminus V(I_i)),\ i = 1, 2.$$

Nonempty implies that there exists $a_i \in X$ and $f_i \in I_i$ s.t. $f_i(a_i) \neq 0$, and hence $f_i \notin P$ for $i = 1, 2$. If $U_1 \cap U_2 = \varnothing$, then

$$X \cap (\mathbb{C}^n \setminus V(I_1)) \cap (\mathbb{C}^n \setminus V(I_2)) = X \cap (\mathbb{C}^n \setminus (V(I_1) \cup V(I_2))) = X \cap (\mathbb{C}^n \setminus V(I_1I_2)) = \varnothing.$$

It implies that $X \subset V(I_1I_2)$ and then $f_1f_2 \in P$, Contradiction. □

Definition 1.5.2. Let S be a ring and $R \subset S$ be a subring. A map $R \to S$ is said to be a **derivation of R (with values in S)** if

(1) $D(x + y) = D(x) + D(y)$, $\forall x, y \in R$,
(2) $D(xy) = xD(y) + yD(x)$,$\forall x, y \in R$.

Definition 1.5.3. Let S be a ring and $R \subset S$ be a subring. Let $R' \subset R$ be a subring. A derivation $D : R \to S$ is called a **R'-derivation** if $D(x) = 0$, $\forall x \in R'$. We denote $\mathcal{D}_{R/R'}(S)$ the set of all R'-derivation of R. If $S = R$, we write $\mathcal{D}_{R/R'} = \mathcal{D}_{R/R'}(S)$

Remark 1.11. (1) $\mathcal{D}_{R/R'}(S)$ is an S-module. In particular, if S is a field, then $\mathcal{D}_{R/R'}(S)$ is an S-vector space.
(2) Assume that R is an integral domain. Let $K = \mathrm{Frac}(R)$. Then any derivation D of R with values in K can be extended uniquely to a derivation of K. Moreover, we have $\mathcal{D}_R(K) \cong \mathcal{D}_K(K)$.

Indeed. Let $x, y \in R$ and $y \neq 0$. Define $D(\frac{x}{y}) := \frac{yD(x) - xD(y)}{y^2}$. Observe that if $\frac{x}{y} = \frac{x'}{y'}$, by definition we have $D(\frac{x}{y}) = D(\frac{x'}{y'})$. It is easy to see that the map $D : K \to K$ is a derivation. Uniqueness is immediately.

Example 1.5.2. (1) *Let R be a ring and D be a derivation on R. Let $A = R[x_1, \ldots, x_n]$. For any*

$$f = \sum c_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

define

$$f^D = \sum D(c_{i_1,\dots,i_n}) x_1^{i_1} \cdots x_n^{i_n}.$$

It gives a derivation of A.

(2) *Let R' be a ring and $R = R'[x_1, \dots, x_n]$. Define*

$$D_i = \frac{\partial}{\partial x_i} : R \to R.$$

with

$$D_i(c) = 0,\ \forall c \in R',$$

$$D_i(\sum c_{k_1,\dots,k_n} x_1^{k_1} \cdots x_n^{k_n}) = \sum c_{k_1,\dots,k_n} k_i x_1^{k_1} \cdots x_n^{k_i - 1}) \cdots x_n^{k_n}).$$

D_i is a R'-derivation.

D_i is uniquelly determinde by (1), (2) in ***Definition*** *1.5.2 and $D_i(c) = 0,\ \forall c \in R'$, $D_i(x_j) = \delta_{ij}$.*

(3) *If $R' = \Bbbk$ is a field and $K = \Bbbk(x_1, \dots, x_n)$. Then $\dim_K \mathcal{D}_{K|\Bbbk} = n$ and $D_1, \dots, D_n$ form a basis for $\mathcal{D}_{K|\Bbbk}$.*

Indeed. Let $D \in \mathcal{D}_{K|\Bbbk}$, we consider $D' := \sum_{i=1}^n D(x_i) D_i \in \mathcal{D}_{K|\Bbbk}$. It is easy to see that $D = D'$. Hence $\mathcal{D}_{K|\Bbbk} = \operatorname{span}(D_1, \dots, D_n)$. It remains to show that $D_1, \dots, D_n$ are linearly independent. Let $\lambda_i \in K$ be such that

$$\sum \lambda_i D_i = 0.$$

Then

$$\lambda_j = (\sum \lambda_i D_i)(x_j) = 0.$$

In fact, we have the following theorems.

Theorem 1.5.1 ([6] Ch.2, Sec.17, Thm41)**.** *Let K be a field,* $\operatorname{char} K = 0$. *Let $F = K(x_1, \dots, x_n)$ by any f.g.extension of K. Then*

$$\operatorname{tr.deg}(F|K) = \dim_F(\mathcal{D}_{F|K}).$$

Corollary 1.5.3 ([6] Ch.2, Sec.17, Cor2$'$)**.** *Let K be a field. Let $F|K$ by a separable algebraic extension. Then any derivation of K can be extended to a derivation of F in a unique way*

Example 1.5.3. *Consider the polynomial ring $K[x_1, \dots, x_n]$ and its field of fraction $F = K(x_1, \dots, x_n)$. Then $\mathcal{D}_{F|K}(F)$ as vector space over F hase basis $D_1, \dots, D_n$.*

Lemma 1.5.3. *There exists a nonempty Zariski open subset $\tilde{U} \subset X$ s.t. $\forall a \in \tilde{U}$, $\dim T_a X = \operatorname{tr.deg}(\mathbb{C}(X)|\mathbb{C})$.*

Proof. Let $\bar{x}_1, \dots, \bar{x}_n$ be the classes of $x_1, \dots, x_n$ in $\mathbb{C}(X)$ Then $\mathbb{C}(X) = \mathbb{C}(\bar{x}_1, \dots, \bar{x}_n)$ and it is f.g.over $\mathbb{C}$. Then by **Theorem** 1.5.1,

$$\operatorname{tr.deg}(\mathbb{C}(X)|\mathbb{C}) = \dim_{\mathbb{C}(X)} \mathcal{D}_{R_X|\mathbb{C}}(\mathbb{C}(X)) = \dim_{\mathbb{C}(X)} \mathcal{D}_{\mathbb{C}[x_1,\dots,x_n]/(P+\mathbb{C})}(\mathbb{C}(X))$$

where

$$\begin{aligned}\mathcal{D}_{\mathbb{C}[x_1,\dots,x_n]/(P+\mathbb{C})}(\mathbb{C}(X)) &= \{(\lambda_1,\dots,\lambda_n)\in\mathbb{C}(X)\mid \sum_{i=1}^n \lambda_i D_i(f)=0,\ \forall f\in P\}\\ &= \{(\lambda_1,\dots,\lambda_n)\in\mathbb{C}(X)\mid \sum_{i=1}^n \lambda_i \frac{\partial}{\partial x_i}(f)=0,\ \forall f\in P\}\end{aligned}$$

And then the dimension of this set is $n-\mathrm{rk}_{\mathbb{C}(X)}(\frac{\partial f_j}{\partial x_i})$.

Claim. There exists a nonempty Zariski open subset $\tilde{U}\subset X$ s.t.

$$\mathrm{rk}_{\mathbb{C}(X)}(\frac{\partial f_j}{\partial x_i}) = \mathrm{rk}_{\mathbb{C}}(\frac{\partial f_j}{\partial x_i}(a)),\ \forall a\in\tilde{U}.$$

$r:=\mathrm{rk}_{\mathbb{C}(X)}(\frac{\partial f_j}{\partial x_i})$

Indeed. By linear algebra we know that there exist $A\in \mathrm{GL}_m(\mathbb{C}(X))$ and $B\in\mathrm{GL}_n(\mathbb{C}(X))$ s.t.

$$A(\frac{\partial f_j}{\partial x_i})B=\begin{pmatrix} I_r & 0\\ 0 & 0\end{pmatrix}$$

But we can write $A=\frac{1}{\alpha}A_0,\ B=\frac{1}{\beta}B_0$ for some $\alpha,\beta\in R_X$ and $A_0\in \mathrm{Mat}_m(R_X)$ and $B_0\in\mathrm{Mat}_n(R_X)$.

Let $\tilde{U}:=\{a\in X\mid \alpha(a)\beta(a)\det(A_0(a))\det(B_0 a)\neq 0\}$, which is a nonempty Zariski open set. And for any $a\in\tilde{U}$,

$$\frac{1}{\alpha(a)}A_0(a)(\frac{\partial f_j}{\partial x_i}(a))\frac{1}{\beta(a)}B_0(a)=\begin{pmatrix} I_r & 0\\ 0 & 0\end{pmatrix}$$

Hence $r=\mathrm{rk}_{\mathbb{C}}(\frac{\partial f_j}{\partial x_i}(a)),\ \forall a\in\tilde{U}$. □

Now we give the proof of **Proposition** 1.5.1.

Proof. We have seen that there exists a nonempty Zariski open subset $U\subset X$ s.t. $\forall a\in U$, $\dim T_aX=\min\{\dim T_bX\mid b\in X\}$. Then by **Lemma** 1.5.2, $U\cap\tilde{U}\neq\varnothing$, where $\tilde{U}$ is as in the **Lemma** 1.5.3. □

1.6. Structure of affine varieties at smooth points.

Theorem 1.6.1 ([5], Thm 1.16, Cor 1.20)**.**

(1) Let $f_1,\dots,f_r\in\mathbb{C}[x_1,\dots,x_n]$ without constant terms ($f_j(0)=0,\ j=1,\dots,r$) and s.t. the linear parts are linearly independent ($\frac{\partial f_j}{\partial x_1}(0),\dots,\frac{\partial f_j}{\partial x_n}(0),\ j=1,\dots,r$, are linearly independent.) Define

$$P:=\{g\in\mathbb{C}[x_1,\dots,x_n]\mid \frac{\sum_{j=1}^r h_jf_j}{K}=g,\ h_j,K\in\mathbb{C}[x_1,\dots,x_n],\ K(0)\neq 0\}.$$

Then P is a prime ideal and $X:=V(P)$ is a variety of dimension $n-r$ and $0\in X$ is a smooth point.

Moreover, $V(f_1,\dots,f_r)=X\cup Y$ where Y is a closed algebraic set s,t, $0\notin Y$.

(2) *Conversely, if $X=V(P)\subset\mathbb{C}^n$ is an affine variety of dimension $n-r$ and $a\in X$ is smooth. Then there exist $f_1,\dots,f_r\in P$ s.t.*

$$\mathrm{rk}(\frac{\partial f_j}{\partial x_i}(a))=r$$

and

$$P=\{g\in\mathbb{C}[x_1,\dots,x_n]\mid \frac{\sum_{j=1}^r h_jf_j}{K}=g,\ h_j,K\in\mathbb{C}[x_1,\dots,x_n],\ K(0)\neq 0\}.$$

Example 1.6.1. *Again consider $X=\{(a^3,a^4,a^5)\mid a\in\mathbb{C}\}=V(P)\subset\mathbb{C}^3$ where $P=(xz-y^2,x^3-yz,x^2y-z^2)$. And it is easy to see that $(1,1,1)\in X$ is a smooth point. One can check that it satisfies (2) in **Theorem** 1.6.1.*

1.7. **The local ring of a point.** Let $R=\mathbb{C}[x_1,\dots,x_n]$, $P=(x_1-a_1,\dots,x_n-a_n)$ where $a=(a_1,\dots,a_n)\in\mathbb{C}^n$. Here P is a maximal ideal. $\mathcal{O}_{\mathbb{C}^n,a}=R_P$ is called the **local ring** a whose elements are rational functions defined in some neighborhood of a.

Remark 1.12. If $g\in\mathbb{C}[x_1,\dots,x_n]$ is s.t. $g(a)\neq 0$, then we can consider $\tilde g(y_1,\dots,y_n):=g(a_1+y_1,\dots,a_n+y_n)\in\mathbb{C}[y_1,\dots,y_n]$ and then $\tilde g(0)\neq 0$. Then it as a inverse in the ring of formal power series

$$\frac{1}{\tilde g(y)}=\sum_{i_1,\dots,i_n=0}^{\infty}c_{i_1,\dots,i_n}y_1^{i_1}\cdots y_n^{i_n}\in\mathbb{C}[[y_1,\dots,y_n]].$$

For example,

$$\frac{1}{1-\sum c_iy_i}=1+\sum_{k=1}^{\infty}\left(\sum_{i=1}^n c_iy_i\right)^k$$

Hence we have $\mathcal{O}_{\mathbb{C}^n,a}\subset\mathbb{C}[[y_1,\dots,y_n]]$. Then in a neighborhood of the smooth point a, it is also a complex manifold in the Euclidean topology.

Now we consider the case of affine variety. Let $X=V(P)\in\mathbb{C}^n$ be an affine variety, $a\in X$. And let $\bar M_a:=(\bar x_1-a_1,\dots,\bar x_n-a_n)$. We can also define $\mathcal{O}_{X,a}:=(R_X)_{\bar M_a}$ the local ring of $a\in X$.

Remark 1.13. Note that $\mathrm{Frac}(\mathcal{O}_{X,a})=\mathbb{C}(X)$.

Proposition 1.7.1. *$R_X=\cap_{a\in X}\mathcal{O}_{X,a}$ in $\mathbb{C}(X)$.*

Proof. $(\subset)$ We have the map $f\mapsto\frac{f}{1}\in\mathcal{O}_{X,a},\ \forall a\in X$.

$(\supset)$ Let $u\in\cap_{a\in X}\mathcal{O}_{X,a}$. Let $I:=\{h\in\mathbb{C}[x_1,\dots,x_n]\mid \bar h u\in R_X\}$ where $\bar h$ is the class of h in R_X. Note that I is an ideal and $P\in I$.

For any $a\in X$, since $u\in\cap_{a\in X}\mathcal{O}_{X,a}$ can be expressed as

$$u=\frac{f}{g},\ g(a)\neq 0.$$

Hence $g \in I$. But $g(a) \neq 0$, if follows that $a \notin V(I)$. And since $P \subset I$, we have $V(I) \subset X$ and $V(I) = \varnothing$. By **Theorem** 1.2.1, $1 \in \sqrt{I}$. Therefore, $1 \in I$ and $u = 1 \cdot u \in R_X$ $\square$

1.8. **Power series expansions.** In this section we introduce the local parameters at a (smooth) point.

Let $X = V(P)$ be an affine variety with $P = (f_1, \ldots, f_m) \subset \mathbb{C}[x_1, \ldots, x_n]$ prime. For $a \in X$, we denote $\bar{M}_a := (\bar{x}_1 - a_1, \ldots, \bar{x}_n - a_n)$ the maximal ideal of a. In the local ring $\mathcal{O}_{X,a} = (R_X)_{\bar{M}_a}$, define $\mathfrak{m}_a := \varphi(\bar{M}_a \cdot \mathcal{O}_{X,a})$ the maximal ideal of $\mathcal{O}_{X,a}$, where $\varphi : R_X \to \mathcal{O}_{X,a}$ with $\varphi(\bar{f}) = \frac{\bar{f}}{1}$.

Let $D \in Der_{R_X,a}$. Then D can be extended to the $Der_{\mathcal{O}_{X,a},a}$ in a unique way by setting $D(\frac{\bar{f}}{\bar{g}}) := \frac{\bar{g}(a)D(\bar{f}) - \bar{f}(a)D(\bar{g})}{\bar{g}(a)^2}$. Note that $D(\mathfrak{m}_a^2) = 0$. D induces a $\mathbb{C}$-linear map $d \in (\mathfrak{m}_a/\mathfrak{m}_a^2)^* : \mathfrak{m}_a/\mathfrak{m}_a^2 \to \mathbb{C}$.

Theorem 1.8.1. *The map*

$$\begin{aligned} Der_{R_X,a} &\to (\mathfrak{m}_a/\mathfrak{m}_a^2)^* \\ D &\mapsto d \end{aligned}$$

is a $\mathbb{C}$-linear isomorphism.

Proof. Linearity is immediately by construction.

(Inj). Let the image of D be 0, then $D|_{\mathfrak{m}_a} = 0$. D induces a $\mathbb{C}$-linear map $\mathcal{O}_{X,a}/\mathfrak{m}_a (\cong \mathbb{C}) \to \mathbb{C}$. By Leibniz, $D = 0$.

(Surj). Let $\delta \in (\mathfrak{m}_a/\mathfrak{m}_a^2)^*$. Define $D : R_X \to \mathbb{C}$ by $D(\bar{f}) := \delta(\varphi(\bar{f} - \bar{f}(a)))$. "standard calculation"

Hence $D \in Der_{R_X,a}$ and its image is δ. $\square$

Remark 1.14. (1) $\forall a \in X$, $\dim_{\mathbb{C}} \mathfrak{m}_a/\mathfrak{m}_a^2 = \dim_{\mathbb{C}} T_a X < \infty$.
(2) From Exer.6, Assignment 1,

$$\begin{aligned} Der_{R_X,a} &\to \mathbb{C}^n \\ D &\mapsto (D(\bar{x}_1), \ldots, D(\bar{x}_n)) \end{aligned}$$

is injective and $\mathbb{C}$-linear. Moreover,

$$D(\bar{f}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a) D(\bar{x}_i).$$

Now identify $D \in Der_{R_X,a}$ with its image $d \in (\mathfrak{m}_a/\mathfrak{m}_a^2)^*$. Let $\bar{u} \in \mathfrak{m}_a/\mathfrak{m}_a^2$. Then

$$d(\bar{u}) = D(u) = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(a) D(\bar{x}_i).$$

where u is any representative of $\bar{u}$ in $\mathfrak{m}_a$.

Definition 1.8.1. Let $r = \dim X$. and Let $a \in X$ be a smooth point. $u_1, \ldots, u_r \in \mathcal{O}_{X,a}$ are called **local parameters** at a if $u_1, \ldots, u_r \in \mathfrak{m}_a$ and their classes $\bar{u}_1, \ldots, \bar{u}_r \in \mathfrak{m}_a/\mathfrak{m}_a^2$ form a basis of $\mathfrak{m}_a/\mathfrak{m}_a^2$. And $\mathfrak{m}_a/\mathfrak{m}_a^2$ is called the **cotangent space** of X at a.

Remark 1.15. If $u_1, \dots, u_r \in \mathcal{O}_{X,a}$ are local parameters at a, then $\mathrm{rk}(\frac{\partial u_j}{\partial x_i}(a)) = r$.

Example 1.8.1. *Let $X = V(x^2 + y^2 - 1) \subset \mathbb{C}^2$, $a = (0, 1)$. Then $T_aX = \{(x, y) \mid y = 1\}$. $x \in \mathfrak{m}_a$ is a local parameter since $dx|_{\mathfrak{m}_a/\mathfrak{m}_a^2}$. $y - 1 \in \mathfrak{m}_a$ is not a local parameter since $D(\bar{y}) = 0$, $\forall D \in Der_{R_X,a}$ and $\frac{\partial y-1}{\partial x}(a)D(\bar{x}) + \frac{\partial y-1}{\partial y}(a)D(\bar{y}) = 0$.*

Lemma 1.8.1 (Nakayama)**.** *Let R be a ring and M be a f.g. R-module. Let $I \subset R$ be an ideal. Then $IM = M \Leftrightarrow \exists x \in 1 + I$ s.t. $xM = 0$.*

In particular, if $\forall x \in 1 + I$ is invertible(e.g. I is maximal), then $IM = M \Leftrightarrow M = 0$.

Proof. $(\Leftarrow)$. Clear.

$(\Rightarrow)$. Let $v_1, \dots, v_n \in M$ be generators of M. $IM = M$ implies that $\exists a_{ij} \in I$, $1 \leq i, j \leq n$ s.t. $-v_i = \sum_j a_{ij} v_j$. Then we have $(A + I_n)v = 0$, where $A = (a_{ij})$, I_n is the identity matrix and $v = (v_1, \dots, v_n)^T$. Then by multplying the classical adjoint matrix of $A+I$(classical adjoint matrix S^* of matrix S is s.t. $SS^* = (\det S)I$ with I the identity). $(A+I_n)^*(A+I_n)v = 0$. But $(A + I_n)^*(A + I_n) = \det(A + I_n)I_n$. And $\det(A + I_n) = 1 + d$ for some $d \in I$. Hence $(1 + d)v_i = 0$, $\forall i = 1, \dots, n$. Then $(1 + d)M = 0$ □

Corollary 1.8.1. *Let R as before. Let $I \subset R$ be an ideal s.t. every element in $1+I$ is invertible. Let M be a f.g. R-module and $M' \subset M$ be a submodule. Then*

$$M = IM \mod M' \iff M' = M.$$

In particular, $v_1, \dots, v_n \in M$ generate $M \iff$ their classes $\bar{v}_1, \dots, \bar{v}_n \in M/IM$ are generators.

Proof. Applying **Lemma** 1.8.1 to M/M'. Note that $M = IM \mod M' \iff IM/M' = M//M'$. By **Lemma** 1.8.1, it is equivalent to $\exists x \in 1 + I$ s.t. $xM/M' = 0$. But x is invertible. Therefore $M/M' = 0$.

For the last statement, let $M' := \mathrm{span}(v_1, \dots, v_n)$. □

Remark 1.16. In particular, if $u_1, \dots, u_r \in \mathfrak{m}_a$ are local parameters, then $\mathfrak{m}_a = (u_1, \dots, u_r)$.

Let $u_1, \dots, u_r \in \mathcal{O}_{X,a}$ be local parameters, where $a \in X$ is a smooth point. For any $v \in \mathcal{O}_{X,a}$, define $v_{(1)} := v - v(a) \in \mathfrak{m}_a$. Then $\exists \lambda_1, \dots, \lambda_r \in \mathbb{C}$ s.t.

$$v_{(1)} - \lambda_1 u_1 - \dots - \lambda_r u_r \in \mathfrak{m}_a^2.$$

Let $v_{(2)} := v_{(1)} - \sum \lambda_i u_i \in \mathfrak{m}_a^2$. We can write $v_{(2)} = \sum w_k x_k$ for some $w_k, x_k \in \mathfrak{m}_a$

Again $\exists \mu_{ki}, \nu_{ki} \in \mathbb{C}$ s.t.

$$w_k - \sum_i \mu_{ki} u_i, x_k - \sum_i \nu_{ki} - u_i \in \mathfrak{m}_a^2.$$

Hence

$$v_{(2)} = \sum_k \left(\sum_i \mu_{ki} u_i + \mathfrak{m}_a^2 \right) \left(\sum_i \nu_{ki} u_i + \mathfrak{m}_a^2 \right)$$
$$= \sum_{k,l} \gamma_{k,l} u_k u_l + \mathfrak{m}_a^3.$$

By repeating this procedure, we can construct homogeneous polynomial $v_{(l)} \in \mathbb{C}[t_1, \ldots, t_r]$ of degree l for any $l \in \mathbb{N}$ s.t.

$$v = \sum_{l=0}^{k} v_{(l)}(u_1, \ldots, u_r) + \mathfrak{m}_a^{k+1}.$$

Definition 1.8.2. The **formal power series ring** in $t = (t_1, \ldots, t_r)$ is the ring $\mathbb{C}[[t]] = \mathbb{C}[[t_1, \ldots, t_r]]$ whose elements are infinite sum of the form

$$\Phi = F_1 + F_2 + \cdots$$

where $F_i \in \mathbb{C}[t_1, \ldots, t_r]$ is homogeneous of degree i.

And the operations are the following. Let $\Psi = \sum H_i$.

$$\Phi + \Psi = \sum_i F_i + H_i$$
$$\Phi \cdot \Psi = \sum_{i \geq 0} (\sum_{j+k=i} H_j F_k)$$

Remark 1.17. (1) We can replace $\mathbb{C}$ by any other field.
(2) $\mathbb{C} \subset \mathbb{C}[[t]]$.
(3) $\mathbb{C}[[t]]$ is an integral domain.

Definition 1.8.3. Let $u \in \mathcal{O}_{X,a}$ with a smooth point. Let $u_1, \ldots, u_r$ be local parameters at a. A **Taylor series** for u is $\Phi = \sum F_i \in \mathbb{C}[[t_1, \ldots, t_r]]$ s.t.

$$u - \sum_{i=0}^{k} F_i(u_1, \ldots, u_r) \in \mathfrak{m}_a^{k+1}, \ \forall k.$$

Example 1.8.2.

(1) $X = \mathbb{C}$, $a = 0$, $\mathfrak{m}_a = (x)$. *Then* $\forall \frac{f}{g} \in \mathcal{O}_{X,a}$ *has a unique Taylor series* $\sum_{i \geq 0} \lambda_i x^i$ *with* $\frac{f}{g} - \sum_{i=0}^{k} \lambda_i x^i \in (x^{k+1})$.
(2) $\frac{1}{1-x} = \sum_{i \geq 0} x^i$.

Theorem 1.8.2 (Weierstrass Preparation Theorem, [7],p139(simplified version))**.** *Let* $\Phi = \sum_{i=1}^{n} a_i x_i +$ *higher order terms. Assume that* $a_1 \neq 0$. *Then for any* $\Psi \in \mathbb{C}[[x_1, \ldots, x_n]]$, *there are unique* $A, B \in \mathbb{C}[[x_1, \ldots, x_n]]$ *s.t.* $\Psi = A\Phi + B$.

Corollary 1.8.2. $\mathbb{C}[[x_1, \ldots, x_n]]/(\Phi) = \mathbb{C}[[x_2, \ldots, x_n]]$

Theorem 1.8.3. *$\forall u \in \mathcal{O}_{X,a}$ with a smooth point. Let $u_1, \ldots, u_r$ be local parameters at a. There exists a unique Taylor series of u. Consequently, we have a injective morphism*

$$\begin{aligned} \tau : \mathcal{O}_{X,a} &\to \mathbb{C}[[t_1, \ldots, t_r]] \\ u &\mapsto \tau(u) \end{aligned}$$

where $\tau(u)$ is the Taylor series of u.

Proof. It is sufficient to prove the following claim.

Claim. If $F_k \in \mathbb{C}[t_1, \ldots, t_r]$ is homogeneous of degree k and $F_k(u_1, \ldots, u_r) \in \mathfrak{m}_a^{k+1}$, then $F_k = 0$.

Indeed. We prove the claim in two steps.

Step 1. Consider the case that $X = \mathbb{C}^n$, $r = n$, $u_1 = x_1 - a_1, \ldots, u_n = x_n - a_n$, $\mathfrak{m}_a = (x_1 - a_1, \ldots, x_n - a_n)$. Then $\mathfrak{m}_a^k = \{\prod_{i=1}^n (x_i - a_i)^{m_i} \mid \sum_{i=1}^n m_i = k\}$.

In this case, if $F_k(x_1 - a_1, \ldots, x_n - a_n) \in \mathfrak{m}_a^{k+1}$, $F_k = 0$.

Step 2. Now we consider the general case that $X = V(P)$ with P prime. Assume, WLOG, that $a = 0$. We have

$\frac{\overline{f}}{\overline{g}} = 0$ iff

Hence the map $\mathcal{O}_{\mathbb{C}^n,0}/P\mathcal{O}_{\mathbb{C}^n,0} \to \mathcal{O}_{X,0}$ is an isomorphism. We have the following commutative diagram.

It easy to see that $\mathcal{O}_{\mathbb{C}^n,0}/P\mathcal{O}_{\mathbb{C}^n,0} \to \mathbb{C}[[x_1, \ldots, x_n]]/P\mathbb{C}[[x_1, \ldots, x_n]]$ is injective. It remains to show that $\mathbb{C}[[x_1, \ldots, x_n]]/P\mathbb{C}[[x_1, \ldots, x_n]] \cong \mathbb{C}[[y_1, \ldots, y_r]]$.

By **Theorem** 1.6.1, there exist $f_1, \ldots, f_{n-r} \in \mathbb{C}[x_1, \ldots, x_n]$ s.t. $P\mathcal{O}_{\mathbb{C}^n,0} = (\frac{f_1}{1}, \ldots, \frac{f_{n-r}}{1})$ and $P\mathbb{C}[[x_1, \ldots, x_n]] = (f_1, \ldots, f_{n-r})\mathbb{C}[[x_1, \ldots, x_n]]$.

If $f_i = \sum_{j=1}^n a_{ij}x_j +$ h.o.t, then $\mathrm{rk}(a_{ij})_{\substack{i=1,\ldots,n-r \\ j=1,\ldots,n}} = \mathrm{rk}(\frac{\partial f_i}{\partial x_j}) = n - r$. And up to a linear change of coordinates we can assume that $\mathrm{rk}(a_{ij})_{1 \le i,j \le n-r} = n - r$. By **Theorem** 1.8.2,

$$\mathbb{C}[[x_1, \ldots, x_n]]/P\mathbb{C}[[x_1, \ldots, x_n]] \cong \mathbb{C}[[x_{n-r+1}, \ldots, x_n]].$$

Consider $u_1, \ldots, u_r \in \mathbb{C}[[y_1, \ldots, y_r]]$, then $u_i = \sum_{j=1}^r b_{ij}y_j$ with $b_{ij} \in \mathbb{C}[[y_1, \ldots, y_r]]$. Set $B := (b_{ij}) \in \mathrm{Mat}_r(\mathbb{C}[[y_1, \ldots, y_r]])$. $\det B(0) \neq 0$ since $u_1, \ldots, u_r$ form a basis of $\mathfrak{m}_0/\mathfrak{m}_0^2$ and also $y_1, \ldots, y_r$ form a basis of $\mathfrak{m}_0/\mathfrak{m}_0^2$. There exists $B^{-1} = \det(B)^{-1}B^*$. Hence $\mathbb{C}[[u_1, \ldots, u_r]] \cong \mathbb{C}[[y_1, \ldots, y_r]])$. The claim follows from Step 1.

Now we have the morphism

$$\begin{aligned} \tau : \mathcal{O}_{X,a} &\to \mathbb{C}[[t_1, \ldots, t_r]] \\ u &\mapsto \tau(u). \end{aligned}$$

Let $u \in \mathcal{O}_{X,a}$ be such that $\tau(u) = 0$. Then $u \in \mathfrak{m}_a^k$, $\forall k$. Hence $u \in \cap_{k \ge 0}\mathfrak{m}_a^k$. But $\mathfrak{m}_a(\cap_{k \ge 0}\mathfrak{m}_a^k) = \cap_{k \ge 0}\mathfrak{m}_a^k$. Then by **Lemma** 1.8.1, $\cap_{k \ge 0}\mathfrak{m}_a^k = 0$, hence $u = 0$. Therefore, the morphism is injective. $\square$

Remark 1.18. (Formal Implicit Function theorem) Let $\Phi = \sum_{i=1}^n a_i x_i +$ higher order terms $\in \mathbb{C}[[x_1, \ldots, x_n]]$ s.t. $a_1 \neq 0$. Then by **Theorem** 1.8.2, there exist $A \in \mathbb{C}[[x_1, \dots, x_n]]$ and $B \in \mathbb{C}[[x_2, \dots, x_n]]$ s.t. $x_1 = A\Phi + B$ Note that $A(0) \neq 0$, then $\Phi = (x_1 - B)A^{-1}$.

1.9. **Analytic structure of smooth point.**

Definition 1.9.1 ([2])**.** A complex manifold of complex dimension n is a Hausdorff and 2nd countable topological space M together with a holomorphic atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$. i.e. $M = \cup_\alpha U_\alpha$ with $U_\alpha \subset M$ open and $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is holomorphic. And a homeomorphism $\phi U_\alpha \to V_\alpha$ for some $V_\alpha \subset \mathbb{C}^n$ open.

Remark 1.19. By $\varphi_\alpha \circ \varphi_\beta^{-1}$ holomorphic we mean that it is C^0 and satisfies the Cauchy-Riemann equations, which is

Theorem 1.9.1. *If $X \subset \mathbb{C}^n$ is an affine variety of dimension r. Let $U \subset X$ be the set of smooth points. Then U is a complex manifold of dimension r.*

1.10. $\mathcal{O}_{X,a}$ **is a UFD.** [2]

Proposition 1.10.1 ([4], Appendix 7)**.** *Suppose R is a Noetherian ring with maximal ideal M. Let $\hat{R} \supset R$ with maximal ideal $\hat{M}$ be a local ring, which is a UFD. If the following conditions are satisfied:*
(1) $M\hat{R} = \hat{R}$,
(2) $(M^n\hat{R}) \cap R = M^n, \ \forall n > 0$,
(3) $\forall x \in \hat{R}$ *and* $\forall n > 0$, $\exists x_n \in R$ *s.t.* $x - x_n \in M^n\hat{R} = \hat{M}^n$,
then R is a UFD.

Theorem 1.10.1. *Let X be an affine variety of dimension r and $a \in X$ be a smooth point. Then $\mathcal{O}_{X,a}$ is a UFD.*

Proof. (Sketch). We have seen that $\mathcal{O}_{X,a} \subset \mathbb{C}[[t_1, \dots, t_r]]$.

We use the fact that $\mathbb{C}[[t_1, \dots, t_r]]$ is a UFD.(see details in [7]). Denote $\hat{\mathcal{O}}_{X,a} := \mathbb{C}[[t_1, \dots, t_r]]$ with maximal ideal $\hat{\mathfrak{m}}_a$.

We prove by check the conditions in **Proposition** 1.10.1.

(1) $(\subset)$ Clear.
$(\supset)$ If $u_1, \dots, u_r \in \mathcal{O}_{X,a}$ are local parameters, then $\tau(u_i) = t_i, \ \forall i = 1, \dots, r$. Let $\Phi \in \hat{\mathfrak{m}}_a$, then
$$\Phi = \sum_{i=1}^{r} \varphi_i t_i, \ \varphi_i \in \hat{\mathcal{O}}_{X,a}.$$
Then $\Phi = \sum_{i=1}^{r} \varphi_i \tau(u_i) \in \mathfrak{m}_a \hat{\mathcal{O}}_{X,a}$
(2) $(\supset)$ Clear.
$(\subset)$ Let $\Phi \in (\mathfrak{m}_a^n \hat{\mathcal{O}}_{X,a}) \cap \mathcal{O}_{X,a} = \hat{\mathfrak{m}}_a^n \mathcal{O}_{X,a}$. Since $\Phi \in R$, $\Phi = \tau(u)$. Since $\Phi \in \hat{\mathfrak{m}}_a^n$, $u \in \mathcal{O}_{X,a}^n$.
(3) Clear.

□

[2]This part is a bit of mess. I will try to rewrite it in a more readable way.

Remark 1.20. In general, R_X is not a UFD. (e.g. $X = V(x^2 + y^2 - 1)$)

Corollary 1.10.1 (Geometric interpretation of UFD)**.** *Let $X = V(P)$ with $P \in \mathbb{C}[x_1, \ldots, x_n]$ prime. Let $r = \dim X$ and let $a \in X$ be a smooth point. If $f \in \mathbb{C}[x_1, \ldots, x_n]$ is such that*

$$\frac{\bar{f}}{1} \in \mathcal{O}_{X,a} \text{ is irreducible,}$$

then

$$P' := \{g \in \mathbb{C}[x_1, \ldots, x_n] \mid kg \in P + (f),\ k\mathbb{C}[x_1, \ldots, x_n],\ k(a) \neq 0 \in\}$$

is a prime ideal and $X' := V(P') \subset X$ is an affine variety of dimension $r - 1$. Such X' is called a subvariety of X of codimension 1. Moreover, every subvariety of X of codimension 1 is of this form. In this situation, $\frac{\bar{f}}{1} \in \mathcal{O}_{X,a}$ is called a local equation of X' at a.

Proof. Let ψ be the composition of the following maps

$$\mathbb{C}[x_1, \ldots, x_n] \to R_X \to \mathcal{O}_{X,a}.$$

quotient map, $\varphi : \bar{h} \mapsto \frac{\bar{h}}{1}$. Then $P' = \psi^{-1}(\frac{\bar{f}}{1})$.

Since $\mathcal{O}_{X,a}$ is a UFD and $\frac{\bar{f}}{1}$ is irreducible, $(\frac{\bar{f}}{1})$ is prime. Hence $\psi^{-1}(\frac{\bar{f}}{1})$ is prime.

Let $X' := V(P')$. Note that $X' \subsetneq X$. So $\dim X' < \dim X = r$. But

$$\begin{aligned} T_aX' &= \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid \sum_{i=1}^{n} \frac{\partial h}{\partial x_i}(a)(x_i - a_i) = 0,\ \forall h \in P'\} \\ &= \{(x_1, \ldots, x_n) \in T_aX \mid \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i) = 0\}. \end{aligned}$$

Therefore $\dim T_aX' \geq \dim T_aX - 1 = r - 1$. We can assume WLOG that a is also a smooth point of X' since smooth points form a Zariski open subset. Then we have

$$r - 1 \leq \dim X' < r.$$

Conversely, let $a \in X'$ be such that a is a smooth point of X. Let $P' = I(X') \subset \mathbb{C}[x_1, \ldots, x_n]$. Then $\psi(P')\mathcal{O}_{X,a} \subset \mathcal{O}_{X,a}$ and it is a prime ideal. Let $\frac{\bar{f}}{\bar{g}} \in \psi(P')\mathcal{O}_{X,a} \setminus \{0\}$, can assume $\frac{\bar{f}}{\bar{g}}$ is irreducible. Consider $\psi^{-1}(\frac{\bar{f}}{1}) \subset \mathbb{C}[x_1, \ldots, x_n]$. Since $\bar{f}$ is irreducible, we have

$$X' \subset V(\psi^{-1}(\frac{\bar{f}}{1})) \subsetneq X,$$

and hence $r-1 \leq \dim V(\psi^{-1}(\frac{\bar{f}}{1})) < r$. Finally we have $X' = V(\psi^{-1}(\frac{\bar{f}}{1}))$. $\square$

Remark 1.21. Consider $X = \mathbb{C}$, then $R_X = \mathbb{C}[X]$. Every $f \in R_X$ is determined (up to scalar multiple) by its zeros.

Can write f as $f = c(x - z_1)^{\mu_1} \cdots (x - z_k)^{\mu_k}$. Then f is determined by $(z_i, \mu_i),\ i = 1, \ldots, k$.

Let $\mathrm{Div}(X)$ be the free abelian group generated by the set of all points of X, which is

$$\mathrm{Div}(X) := \{f : X \to \mathbb{Z} \mid |\operatorname{supp} f| < \infty\}.$$

The elements have the form of formal sum $\sum_{i=1}^{k} \mu_i z_i \leftrightarrow f : X \to \mathbb{Z}$, $f(x) = \mu_i$ if $x = z_i$.

Moreover, if $\frac{f}{g} \in \mathbb{C}(x) = \mathbb{C}(X)$ with

$$f = c \prod_{i=1}^{k} (x - z_i)^{\mu_i},\ z_i \neq z_j \text{ if } i \neq j,$$
$$g = c' \prod_{j=1}^{m} (x - w_j)^{\nu_j},\ w_j \neq w_l \text{ if } j \neq l,$$

then

$$\frac{f}{g} \mapsto \sum_{i=1}^{k} \mu_i z_i - \sum_{j=1}^{m} \nu_j w_j.$$

It is a bijection if R_X is a UFD.

Now we consider a more general case.

Definition 1.10.1. Let X be any affine variety. Let $\mathcal{Z}$ be the set of subvarieties of X of codimension 1. We define $\mathrm{Div}(X)$ as the free abelian group generated by $\mathcal{Z}$. ($\mathrm{Div}(X) := \{f, \mathcal{Z} \to \mathbb{Z} \mid f(Z) \neq 0,$ for finitely many $Z \in \mathcal{Z}\}$)

$\mathrm{Div}(X)$ is called the **group of divisors** of X.

Remark 1.22. Note that any $f \in \mathrm{Div}(X)$ can be written as the formal sum $\sum_{Z \in \mathcal{Z}} f(Z) Z$. Conversely, any formal sum $\sum_{i=1}^{N} \mu_i Z_i$ correspondes to such an $f \in \mathrm{Div}(X)$.

Elements $Z \in \mathcal{Z}$ are said to be prime divisors.

Now assume that X is smooth. Let $f \in R_X \setminus 0$ and $Z \in \mathcal{Z}$. For $a \in Z$, consider a local equation $\frac{h}{1} \in \mathcal{O}_{X,a}$ of Z at a. Consider $\frac{f}{1} \in \mathcal{O}_{X,a}$. Define $\mathrm{ord}_Z(f) := \mu$ s.t. $(\frac{h}{1})^\mu | \frac{f}{1}$ but $(\frac{h}{1})^{\mu+1} \nmid \frac{f}{1}$. Note that $\mathrm{ord}_Z(f)$ doesnot depend on $a \in Z$. In fact, $\mathrm{ord}_Z(f)$ does not depend on $a \in Z$.

So we get a map

$$f \mapsto \mathrm{div}(f) := \sum_{Z \in \mathcal{Z}} \mathrm{ord}_Z(f) Z \in \mathrm{Div}(X).$$

More generally, $\forall \frac{\bar{f}}{\bar{g}} \in \mathbb{C}(X)$, we can define

$$\mathrm{div}(\frac{\bar{f}}{\bar{g}}) := \mathrm{div}(\bar{f}) - \mathrm{div}(\bar{g}) \in \mathrm{Div}(X)$$

and such element is called the **principal divisor**.

Definition 1.10.2. Consider $\mathrm{div}(\frac{\bar{f}}{\bar{g}}) = \sum_{Z \in \mathcal{Z}} n_Z Z \in \mathrm{Div}(X)$. If $n_Z > 0$, we call Z a **zero** for $\frac{\bar{f}}{\bar{g}}$. If $n_Z < 0$, we call Z a **pole** for $\frac{\bar{f}}{\bar{g}}$.

Remark 1.23. Let $P(X) :=$ the principal divisors. Note that $P(X) \subset \mathrm{Div}(X)$ is a subgroup.

Definition 1.10.3. Let X be a smooth affine variety. The quotient group

$$\mathrm{Cl}(X) := \mathrm{Div}(X)/P(X)$$

is called the **divisor class group** or **Picard group** of X.

Remark 1.24. One can prove that in this smooth case, the general definition[3] of Picard group is the same as the definition above.

Proposition 1.10.2. $\mathrm{Cl}(X) = \{0\} \iff R_X$ *is UFD.*

Remark 1.25. Compare with $X = \mathbb{C}$, we have

$$D = \sum_{z\in\mathbb{C}} n_z z = \mathrm{div}\left(\frac{\prod_{\substack{z\in\mathbb{C}\\ n_z>0}} (x-z)^{n_z}}{\prod_{\substack{z\in\mathbb{C}\\ n_z<0}} (x-z)^{-n_z}}\right).$$

1.11. Morphisms between affine varieties.

Definition 1.11.1. Let $X = V(P) \in \mathbb{C}^n$ and $Y = V(Q) \subset \mathbb{C}^m$ be two affine varieties. A **regular map** or **morphism** between X and Y is a function

$$\varphi = (\varphi_1, \dots, \varphi_m) : X \to Y$$

s.t. with $\varphi_j : X \to \mathbb{C}$ in R_X (i.e. $\varphi_1, \dots, \varphi_m$ are restrictions to X of polynomials in $\mathbb{C}[x_1, \dots, x_n]$).

Definition 1.11.2. Let $\varphi = (\varphi_1, \dots, \varphi_m) : X \to Y$ be a regular map and $a \in X$, we have the differential of φ at a

$$d_a\varphi : T_aX \to T_{\varphi(a)}Y$$
$$(\xi_1, \dots, \xi_n) \mapsto (\sum_{i=1}^n \frac{\partial \varphi_1}{\partial x_i}(a)(\xi_i - a_i) + \varphi_1(a), \dots, \sum_{i=1}^n \frac{\partial \varphi_m}{\partial x_i}(a)(\xi_i - a_i) + \varphi_m(a)).$$

Remark 1.26. (Exer.) One can check that the differential is well defined by using chain rule.

Let $\varphi = (\varphi_1, \dots, \varphi_m) : X \to Y$ be a regular map. It induces a ring homomorphism

$$\varphi^* : R_Y \to R_X$$
$$\bar{f} \mapsto \bar{f} \circ \varphi.$$

Definition 1.11.3. A regular map $\varphi = (\varphi_1, \dots, \varphi_m) : X \to Y$ is called **dominant** if $Y = \overline{\varphi(X)}$.

Proposition 1.11.1. *Let $\varphi : X \to Y$ be a dominant regular map between affine varieties. Then $\varphi^* : R_Y \to R_X$ induces a homomorphism*

$$\varphi^* : \mathbb{C}(Y) \to \mathbb{C}(X).$$

In particular, $\dim X \geq \dim Y$.

[3] isomorphism classes of line bundles(or invertible sheaves)

Proof. Note that if $g \in R_Y \setminus 0$, then $\varphi^*(g) \neq 0$, otherwise $\varphi(X) \subset \{b \in Y \mid g(b) = 0\} \subsetneqq Y$. Then $\overline{\varphi(X)} \neq Y$, contradiction. Hence we can define $\varphi^*(\frac{f}{g}) := \frac{f \circ \varphi}{g \circ \varphi} \in \mathbb{C}(X))$ and it is a homomorphism. $\square$

1.12. **Appendix. primary decomposition.** Now We recall some commutative algebra [6].

Definition 1.12.1. Let R be a ring and $I \subset R$ be an ideal of R. I is called **primary** if whenever $a, b \in R$ are such that $ab \in I$ and $a \notin I$, then $b \in \sqrt{I}$.

We have immediately that the radical of a primary ideal is prime.

Theorem 1.12.1 (Lasker-Noether decomposition theorem)**.**

(1) *Let R be a Noetherian ring, then every ideal $I \subset R$ admits the so called primary representation as*

$$I = Q_1 \cap \cdots \cap Q_N$$

where Q_i's are primary ideals of R.

Moreover, we can find $Q_1, \ldots, Q_N$ s.t.no Q_i contains $\bigcap_{j \neq i} Q_j$ and the associated prime ideals $\sqrt{Q_1}, \ldots, \sqrt{Q_n}$ are distinct. In this case it is called irredundant primary representation.

(2) *Let R be a ring and $I \subset R$ be an ideal that admits an irredundant primary representation*

$$I = Q_1 \cap \cdots \cap Q_N.$$

Then $I = \sqrt{I}$ iff $Q_1, \ldots, Q_N$ are prime.

Theorem 1.12.2. *Let R be a ring and $I \subset R$ be an ideal admitting an irreduandant primary representation*

$$I = Q_1 \cap \cdots \cap Q_N.$$

Then the prime ideals $P_i := \sqrt{Q_i}$ are uniquely determined by I. And they are called the associated primes of I.

Example 1.12.1. *Let $I = (x^2, y) \subset \mathbb{C}[x, y]$ be an ideal. It has an irreduandant primary representation*

$$I = (x^2) \cap (y).$$

And its radical is

$$\sqrt{I} = (x) \cap (y).$$

More generally, let $f \in \mathbb{C}[x_1, \ldots, x_n]$ and write $f = g_1^{k_1} \cdots g_N^{k_N}$ where g_i are irrd and not associated to each other. Then we have

$$(f) = (g_1^{k_1}) \cap \cdots \cap (g_N^{k_N}).$$

And its radical is

$$\sqrt{(f)} = (g_1) \cap \cdots \cap (g_N) = (g_1 \cdots g_N).$$

Example 1.12.2. *Let $\Bbbk$ be any field. Consider the polynomial ring $\Bbbk[x, y]$ and ideal $I = (x^2, xy)$. Then for any $c \in \Bbbk$,*

$$I = (x) \cap (y - cx, x^2)$$

is an irredundant primary representation of I.

Question 1.12.1. *What are the associated primes of I?*

Corollary 1.12.1. *Let $I \subset \mathbb{C}[x_1 \dots, x_n]$ be a radical ideal. Then there exists unique prime ideals $P_1, \dots, P_N \subset \mathbb{C}[x_1 \dots, x_n]$ s.t.*

$$I = P_1 \cap \cdots \cap P_N$$

and $P_i \neq P_j, \ \forall i \neq j$.

1.13. **Appendix. transcendental extension.**

Definition 1.13.1. An extension $K|\Bbbk$ is **transcendental** if it is not algebraic(i.e. if $\exists \alpha \in K$ not algebraic over $\Bbbk$).

Example 1.13.1. (1) $\mathbb{Q}(\pi)|\mathbb{Q}$ *is transcendental.*
(2) $\mathbb{Q}(i)|\mathbb{Q}$ *is algebraic.*
(3) *Let $\Bbbk$ be any field and K be the fraction field of $\Bbbk[x_1, \dots, x_n]$, which is $K = \Bbbk(x_1, \dots, x_n)$. Then $K|\Bbbk$ is transcendental.*

Definition 1.13.2. Let $K|\Bbbk$ be a field extension. Let $L \subset K$. The elements of L are said to be **algebraically independent** over $\Bbbk$ if $\forall \alpha_1, \dots, \alpha_N \in L$, there is no $f \in \Bbbk[x_1, \dots, x_n]$ s.t. $f(\alpha_1, \dots, \alpha_N) = 0$. In this case, L is called a **transcendental set** over $\Bbbk$.

Definition 1.13.3. A **transcendental basis** for $K|\Bbbk$ is a transcendental set $L \subset K$ over $\Bbbk$ that is not contained in any bigger transcendental set.

Remark 1.27. $L \subset K$ is a transcendental basis for $K|\Bbbk$ iff $K|\Bbbk(L)$ is algebraic.

Example 1.13.2. $\{x_1, \dots, x_n\} \in \Bbbk(x_1, \dots, x_n)$ *form a transcendental basis for* $\Bbbk(x_1, \dots, x_n)|\Bbbk$.

Theorem 1.13.1. *There exists a transcendental basis for any field extension. Moreover, any two transcendental basis have the same cardinality.*

See Chapter II Sec.12 in [6] for the proof.

Definition 1.13.4. The cardinality of any transcendental basis for $K|\Bbbk$ is called the **transcendental degree** of $K|\Bbbk$, denoted by $\operatorname{tr.deg}(K|\Bbbk)$.

Remark 1.28. $\operatorname{tr.deg} \mathbb{R}|\mathbb{Q} = \operatorname{tr.deg} \mathbb{C}|\mathbb{Q} = \infty$

1.14. **Appendix. Localization.**

Definition 1.14.1. Let R be a ring and $P \subset R$ be a prime ideal. The **localization** of R at P is

$$R_P := \{(f, g) \in R \times R \mid g \notin P\}/$$

where (f, g) (f', g') iff $\exists h \notin P$ s.t. $(fg' - gf')h = 0$.

One may view the element $(f, g) \in R_P$ as "$\frac{f}{g}$".
We have a morphism

$$\varphi : R \to R_P$$
$$f \mapsto \frac{f}{1}.$$

And $\forall f \in R \setminus P$, $\varphi(f)$ is invertible. More generally, $\frac{f}{g}$ is invertible in R_P if $f \notin P$ and $(\frac{f}{g})^{-1} = \frac{g}{f}$.

Let $\mathfrak{m} := \{\frac{f}{g} \mid f \in P\}$. It is a (unique) maximal ideal of R_P and $(R_P, \mathfrak{m})$ is a local ring.

Remark 1.29. If R is an integral domain then so is R_P.

Proposition 1.14.1. *If R is Noetherian, then so is R_P.*

Proof. Let $I \subset R_P$ be an ideal and $\bar{I} := \varphi^{-1}(I) \subset R$. Since R is Noetherian, $\bar{I} = (\bar{f}_1, \ldots, \bar{f}_m)$ for some $\bar{f}_i \in R$.

Let $u \in I$, then $u = \frac{f}{g}$ and $gu = f \in I$. Then $gu = \varphi(f)$. Hence $f \in \bar{I}$. It follows that $f = \sum h_i \bar{f}_i$. Then $gu = \sum \varphi(h_i)\varphi(\bar{f}_i)$. Hence $u \in (\varphi(\bar{f}_1), \ldots, \varphi(\bar{f}_m))$. □

2. Projective Varieties

2.1. **Projective Space.** We first define the projective space in our setting.

Definition 2.1.1. We define the complex projective n-space by $\mathbb{P}^n := \mathbb{C}^{n+1} \setminus \{0\}/\sim$, where $(a_0, \ldots, a_n) \sim \lambda(a_0, \ldots, a_n)$ for some $\lambda \in \mathbb{C}^\times$

For $(a_0, \ldots, a_n) \in \mathbb{C}^{n+1}$, we denote its equivalence class in $\mathbb{P}^n$ by $[a_0 : \cdots : a_n]$ and it is called the homogeneous coordinates of the cooresponding point.

Remark 2.1. There are also several other equivalent definitions, e.g.

$$\mathbb{P}^n = \{L \subset \mathbb{C}^{n+1} \mid \dim_{\mathbb{C}} L = 1\}.$$

Remark 2.2. Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ by the quotient map, $\pi(a_0, \ldots, a_n) = [a_0 : \cdots : a_n]$. The Euclidean topology induces the quotient topology on $\mathbb{P}^n$, called the classical topology.

$\mathbb{P}^n$ is compact w.r.t. the classical topology and moreover, it is Hausdorff.

One can view $\mathbb{P}^n$ as the compactification of $\mathbb{C}^n$.

For any $i = 0, \ldots, n$, define $H_i := \{[a_0 : \cdots : a_n] \mid a_i = 0\}$ and $U_i := \mathbb{P}^n \setminus H_i$. Then consider the map

$$\begin{aligned} \phi_i : U_i &\to \mathbb{C}^n \\ [a_0 : \cdots : a_n] &\mapsto (\frac{a_0}{a_i}, \ldots, \frac{a_{n-1}}{a_i}, \frac{a_{n+1}}{a_i}, \ldots, \frac{a_n}{a_i}). \end{aligned}$$

It is easy to see that ϕ_i is bijective with inverse

$$\begin{aligned} \phi_i^{-1} : \mathbb{C}^n &\to U_i \\ (z_0, \ldots, z_n) &\mapsto [z_1 : \cdots : z_i : 1 : z_{i+1} : \cdots : z_n] \end{aligned}$$

Remark 2.3. ϕ, ϕ^{-1} are continuous w.r.t. the classical topology. And $\mathcal{A} := \{(U_i, \phi_i) \mid i = 0, \ldots, n\}$ is a topological atlas of $\mathbb{P}^n$.

If $j < i$, $\phi_j \circ \phi i^{-1}$ is holomorphic on $\mathbb{C}^n \setminus \{z_{j+1} = 0\} = \phi_j(U_i \cap U_j)$. In particular, $\mathcal{A}$ is a holomorphic atlas and then $\mathbb{P}^n$ is a complex manifold.

It follows that $\mathbb{P}^n = \mathbb{C}^n \sqcup H_i = \mathbb{C}^n \sqcup \mathbb{P}^{n-1} = \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \cdots \sqcup \mathbb{C}^0$.

Remark 2.4. H_i is called "the point at ∞".

Definition 2.1.2. A **closed algebraic set** in $\mathbb{P}^n$ is a subset $\mathbb{P}^n$ of the form

$$V(f_1, \ldots, f_m) = \{[a_0 : \cdots : a_n] \mid f_j(a_0, \ldots, a_n) = 0,\ j = 1, \ldots, m\}$$

where $f_1, \ldots, f_m \in \mathbb{C}[x_0, \ldots, x_n]$ are homogeneous.

Remark 2.5. We could have required every $f \in \mathbb{C}[x_0, \ldots, x_n]$(not necessarily homogeneous) to be

$$f(\lambda a_0, \ldots, \lambda a_n) = 0,\ \forall \lambda \in \mathbb{C}^\times.$$

We can write $f = \sum_k f_{(k)}$ where $f_{(k)}$ are the homogeneous components of degree k. Then the condition above is equivalent to

$$f_{(k)}(a_0, \ldots, a_n) = 0,\ \forall k.$$

Proposition 2.1.1. *Let $f_1, \ldots, f_m \in \mathbb{C}[x_0, \ldots, x_n]$ be homogeneous. Let $I = (f_1, \ldots, f_m) \subset \mathbb{C}[x_0, \ldots, x_n]$. Then I is an homogeneous ideal(i.e. $\forall f \in I$, its homogeneous components $f_{(k)} \in I$).*

Conversely, if $I \subset \mathbb{C}[x_0, \ldots, x_n]$ is a homogeneous ideal, then there exist $f_1, \ldots, f_m \in \mathbb{C}[x_0, \ldots, x_n]$ homogeneous s.t. $I = (f_1, \ldots, f_m)$.

Proof. Let $g = \sum_{i=1}^m h_i f_i \in I$. Then the homogeneous components of g are

$$g_{(k)} = \sum_{i=1}^m h_{(k-\deg f_i)} f_i \in I,\ \forall k.$$

Conversely, if I is homogeneous. Take generators $I = (f_1, \ldots, f_m)$, then $(f_i)_{(k)} \in I,\ \forall i, k$. Then $I = ((f_1)_{(k)}, \ldots, (f_m)_{(k)} \mid \forall k \geq 0)$. □

Lemma 2.1.1. *Let $I \subset \mathbb{C}[x_0, \ldots, x_n]$ be an ideal. For any $\lambda \in \mathbb{C}^\times$, set $I^\lambda := \{f^\lambda = f(\lambda x_0, \ldots, \lambda x_n) \mid f \in I\}$. Then*

$$I \text{ is homogeneous} \iff I = I^\lambda,\ \forall \lambda \in \mathbb{C}^\times.$$

Proof. ($\Rightarrow$) ($\subset$) Write $f = (f^{\frac{1}{\lambda}})^\lambda$. If I is homogeneous, then $f^{\frac{1}{\lambda}} \in I,\ \forall f \in I,\ \lambda$.

($\supset$) Let $f^\lambda \in I^\lambda,\ f \in I$. Write $f = \sum f_{(k)}$. Since I is homogeneous, $f_{(k)} \in I$. Then $\lambda^k f_{(k)} \in I$. It follows that $f^\lambda = \sum \lambda^k f_{(k)} \in I$.

($\Leftarrow$) Let $f \in I$. Write $f = \sum f_{(k)}$. Then $f^\lambda = \sum \lambda^k f_{(k)} \in I$. Let $d := \deg f = \max\{k \mid f_{(k)} \neq 0\}$. Let $\lambda_0, \ldots, \lambda_d \in \mathbb{C}^\times$ be such that

$$\prod_{i<j}(\lambda_j - \lambda_i) = \begin{pmatrix} 1 & \lambda_0 & \cdots & \lambda_0^d \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_d & \cdots & \lambda_d^d \end{pmatrix} \neq 0$$

Then we have

$$\begin{pmatrix} f^{\lambda_0} \\ \vdots \\ f^{\lambda_d} \end{pmatrix} = \begin{pmatrix} 1 & \lambda_0 & \cdots & \lambda_0^d \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_d & \cdots & \lambda_d^d \end{pmatrix} \begin{pmatrix} f_{(0)} \\ \vdots \\ f_{(d)} \end{pmatrix} \in I^{\oplus d+1}$$

.

Therefore

$$\begin{pmatrix} f_{(0)} \\ \vdots \\ f_{(d)} \end{pmatrix} = \begin{pmatrix} 1 & \lambda_0 & \cdots & \lambda_0^d \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_d & \cdots & \lambda_d^d \end{pmatrix}^{-1} \begin{pmatrix} f^{\lambda_0} \\ \vdots \\ f^{\lambda_d} \end{pmatrix} \in I^{\oplus d+1}$$

and $f_{(k)} \in I,\ \forall k$.

□

Proposition 2.1.2. *Let $I \subset \mathbb{C}[x_0, \ldots, x_n]$ be a homogeneous ideal. Let $\sqrt{I} = P_1 \cap \cdots \cap P_N$ be the irredundant primary representation. Then $\sqrt{I}, P_1, \ldots, P_N$ are homogeneous.*

Proof. By **Lemma** 2.1.1, if I is homogeneous, then $I = I^\lambda,\ \forall \lambda$. Hence $\sqrt{I} = \sqrt{I^\lambda}$. We have that $\sqrt{I}$ is homogeneous.

Moreover, $\sqrt{I^\lambda} = \sqrt{I} = P_1 \cap \cdots \cap P_N$ is an irredundant primary representation of $\sqrt{I^\lambda}$. Hence

$$(\sqrt{I})^\lambda = P_1^\lambda \cap \cdots \cap P_N^\lambda,\ \forall \lambda.$$

And for any $i = 1, ..., N$ there is a j such that $P_i^\lambda = P_j$ for infinitely many $\{\lambda_i\}_{i \geq 1}$. Then $P_i^{\lambda_k \lambda_1^{-1}} = P_i$ for every k.

Hence $\sqrt{I}, P_i$ are all homogeneous. □

2.2. **Projective Varieties.**

We have the following results which are similar with the affine case.

Proposition 2.2.1. *Let $I_1, I_2, I_\alpha \subset \mathbb{C}[x_0, \ldots, x_n],\ \alpha \in A$ be homogeneous ideals. Then*

(1) *If $I_1 \subset I_2$, then $V(I_2) \subset V(I_1)$,*
(2) *$V(I_1) \cup V(I_2) = V(I_1 \cap I_2) = V(I_1 \cdot I_2)$ $(I_1 I_2 = \{fg | f \in I_1, g \in I_2\})$,*
(3) *$V(\sum_{\alpha \in A} I_\alpha) = \bigcap_{\alpha \in A} V(I_\alpha)$,*
(4) *$V(\sqrt{I}) = V(I)$*
$(\sqrt{I} = \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid f^K \in I$ for some $K > 0\})$.

Remark 2.6. From **Proposition** 2.2.1, if $\sqrt{I} = P_1 \cap \cdots \cap P_N$ is an irredundant primary representation, then

$$V(\sqrt{I}) = V(P_1) \cup \cdots \cup V(P_N).$$

Since $P_i \neq P_j$ if $i \neq j$, $V(P_i) \neq V(P_j)$.

Definition 2.2.1. The Zariski topology in $\mathbb{P}^n$ is the topology whose closed subsets are the closed algebraic set in $\mathbb{P}^n$.

Definition 2.2.2. A **projective variety** is an non-empty set $X \subset \mathbb{P}^n$ of the form $X = V(P)$ for some homogeneous prime ideal $P \subset \mathbb{C}[x_0, \dots, x_n]$.

In this case, $R_X : \mathbb{C}[x_0, \dots, x_n]/P$ is the **homogeneous coordinate ring** of X.

Remark 2.7. Let $f \in \mathbb{C}[x_0, \dots, x_n]$ be homogeneous. In general, it doesn't define a "function" $X \to \mathbb{C}$.

Remark 2.8. If $I \subsetneqq \mathbb{C}[x_0, \dots, x_n]$ is a homogeneous ideal, then $I \subset (x_0, \dots, x_n)$.

Note that $V(x_0, \dots, x_n) = \varnothing$.

Theorem 2.2.1 (Hilbert's Nullstellensatz)**.** *Let $I \subset \mathbb{C}[x_0, \dots, x_n]$ be a homogeneous ideal. Let $f \in \mathbb{C}[x_0, \dots, x_n]$ be homogeneous of degree $\deg f \geq 1$. If $f(a) = 0,\ \forall a \in V(I)$, then $f \in \sqrt{I}$.*

Proof. **Case I.** If $V(I) = \varnothing$, then $\{a \in \mathbb{C}^{n+1} \mid g(a) = 0,\ \forall g \in I\}$ is either empty or $\{0\}$. If it is empty, then $I = \mathbb{C}[x_0, \dots, x_n]$ and by **Theorem** 1.2.1, $f \in \sqrt{I}$. If it is $\{0\}$, then again by **Theorem** 1.2.1, $\sqrt{I} = (x_0, \dots, x_n)$. Since $\deg f \geq 1$, $f \in \sqrt{I}$.

Case II. If $V(I) \neq \varnothing$, then $\{a \in \mathbb{C}^{n+1} \mid g(a) = 0,\ \forall g \in I\} = \pi^{-1}(V(I)) \cup \{0\}$ where $\pi : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{P}^n$ is the quotient map. Then by **Theorem** 1.2.1, $f(a) = 0,\ \forall a \in \pi^{-1}(V(I)) \cup \{0\}$ implies that $f \in \sqrt{I}$. ☐

Remark 2.9. We don't have $I(V(J)) = \sqrt{J}$ in the projective setting.

Remark 2.10. Let $I \subset \mathbb{C}[x_0, \dots, x_n]$ be a homogeneous ideal. The set $C := \{a \in \mathbb{C}^{n+1} \mid g(a) = 0,\ \forall g \in I\}$ is a cone and it is called the **affine cone** of $V(I)$.

Corollary 2.2.1. *Let $I \subset \mathbb{C}[x_0, \dots, x_n]$ be a homogeneous ideal. Then $V(I) = \varnothing$ (in $\mathbb{P}^n$) $\iff \sqrt{I} = \mathbb{C}[x_0, \dots, x_n]$ or $\sqrt{I} = (x_0, \dots, x_n)$*

Proposition 2.2.2. *For $i = 0, \dots, n$, let*

$$\phi_i : U_i = \mathbb{P}^n \setminus H_i \to \mathbb{C}^n$$

$$[a_0 : \dots : a_n] \mapsto (\frac{a_0}{a_i}, \dots, \widehat{\frac{a_i}{a_i}}, \dots, \frac{a_n}{a_i}).$$

Then ϕ_i is an homeomorphism w.r.t the Zariski topology.

Moreover,

(1) *If $P \subset \mathbb{C}[x_0, \dots, x_n]$ is a homogeneous prime ideal, $X = V(P)$, then $\phi_i(X \setminus (X \cap H_i)) = \phi_i(X \cap U_i) = V(P')$ with $P' = \{f(y_1, \dots, 1 i-th, \dots y_n) \mid f \in P\} \subset \mathbb{C}[y_1, \dots, y_n]$ is an affine variety.*
(2) *Conversely, if $Q' \subset \mathbb{C}[y_1, \dots, y_n]$ is a prime ideal, then*

$$\overline{\phi_i^{-1}(V(Q'))} = V(Q)$$

where Q is generated by the set $\{x_i^d g(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i}) \mid g \in Q', d = \deg g\}$.

Proof. Consider the case $i = 0$ (other cases are similar). Consider the maps α, β.

Defined by

$$\alpha : f(x_0, \dots, x_n) \mapsto f(1, y_1, \dots, y_n)$$
$$\beta : g \mapsto x_0^d g(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})$$

Note that $\beta(g)$ is homogeneous of degree d. $\alpha(\beta(g)) = g$. But $\beta(\alpha(f)) \neq f$. However, if f is homogeneous and $x_0 \nmid f$ then $\beta(\alpha(f)) = f$.

Let $Y \subset \mathbb{P}^n \setminus H_0 = U_0$ be a Zariski closed subset. Let $\overline{Y}$ be the Zariski closure of Y in $\mathbb{P}^n$. Then

$$\overline{Y} = V(f_1, \dots, f_m)$$

for some homogeneous $f_1, \dots, f_m$. And since $Y = \overline{Y} \cap U_0$,

$$\phi_0(Y) = V(\alpha(f_1), \dots, \alpha(f_m))$$

Hence ϕ_0 is closed.

Conversely, If $Z = V(g_1, \dots, g_m) \subset \mathbb{C}^n$ is a closed algebraic set. Then

$$\phi_0^{-1}(Z) = V(\beta(g_1), \dots, \beta(g_m)) \cap U_0.$$

Hence ϕ_0 is continuous.

(1) Let $P \subset \mathbb{C}[x_0, \dots, x_n]$ be a homogeneous prime ideal. $X = V(P) \subset \mathbb{P}^n$ is a projective variety. We have seen that

$$\phi_0(X \cap U_0) = V(P') \subset \mathbb{C}^n$$

where $P' = \alpha(P) \subset \mathbb{C}[y_1, \dots, y_n]$.

Claim. P' is a prime ideal.

Indeed.

WLOG we can assume that $x_0 \notin P$. Otherwise $X = V(P) \subset H_0$, i.e. $1 \in \alpha(P)$ and then $P' = \mathbb{C}[y_1, \dots, y_n]$.

Note that $\beta(\alpha(f)) \in P$, $\forall f \in P$. Indeed, If $f = x_0^m \tilde{f}$ with $x_0 \nmid \tilde{f}$, then $\beta(\alpha(f)) = \beta(\alpha(\tilde{f}))$. Since $x_0 \notin P$, $\tilde{f} \in P$. Now it is sufficient to consider the case where $x_0 \nmid f$.

Write $f = \sum_{k=0}^d f_{(k)}$ where $d = \deg f$. Since P is homogeneous, $f_{(k)} \in P$. Then

$$\alpha(f) = \sum_k \alpha(f_{(k)})$$
$$\beta(\alpha(f)) = x_0^d \sum_k \alpha(f_{(k)})(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}) = \sum_k x_0^{d-e_k} \beta(\alpha(f_{(k)}))$$

where $e_k = \deg \alpha(f_{(k)})$.

Note that $\beta(\alpha(f_{(k)})) = \frac{f_{(k)}}{x_0^\varepsilon}$ where ε is the maximal power s.t. $x_0^\varepsilon | f_{(k)}$. But since $f_{(k)} \in P$ and $x_0 \notin P$, $\frac{f_{(k)}}{x_0^\varepsilon} \in P$ implies that $\beta(\alpha(f_{(k)})) \in P$, $\forall k$. Hence $\beta(\alpha(f)) \in P$

Now let $g, h \in \mathbb{C}[y_1, \dots, y_n]$ be such that $gh \in P'$. Then there exists $f \in P$ s.t. $gh = \alpha(f)$. Then $\beta(g)\beta(h) = \beta(gh) = \beta(\alpha(f)) \in P$. Since P is prime, either $\beta(g) \in P$ or $\beta(h) \in P$. Hence either $g \in P'$ or $h \in P'$.

(2) Similar. □

Corollary 2.2.2. *If $X \subset \mathbb{P}^n$ is a projective variety. $U_i := \mathbb{P}^n \setminus H_i$, $i = 0, \dots, n$. Then*

$$X = \bigcup_{i=0}^{n} (X \cap U_i).$$

Example 2.2.1. *Conseider $X = V(xy - z^2) \subset \mathbb{P}^2$.*

In $\mathbb{P}^2 \setminus V(x)$, $X \setminus (X \cap V_x) \cong V(u - v^2) \subset \mathbb{C}^2$. And $X \cap V(x) = \{[0 : 1 : 0]\}$

In $\mathbb{P}^2 \setminus V(z)$, $X \setminus (X \cap V_z) \cong V(st - 1) \subset \mathbb{C}^2$. And $X \cap V(z) = \{[0 : 1 : 0], [1 : 0 : 0]\}$.

Definition 2.2.3. Let $X = V(P) \subset \mathbb{P}^n$ be a projective variety.

(1) For any $a = [a_0 :, \cdots : a_n] \in X$, the **local ring** at a in X is

$$\mathcal{O}_{X,a} := \frac{\{\frac{f}{g} \mid f, g \in \mathbb{C}[x_0, \dots, x_n] \text{ homogeneous of the same degree, } g(a) \neq 0\}}{\{\frac{f}{g} \mid f, g \in \mathbb{C}[x_0, \dots, x_n] \text{ homogeneous of the same degree, } g(a) \neq 0,\ f \in P\}}.$$

(2) The **function field** of X is defined as

$$\mathbb{C}(X) := \frac{\{\frac{f}{g} \mid f, g \in \mathbb{C}[x_0, \dots, x_n] \text{ homogeneous of the same degree, } g \notin P\}}{\{\frac{f}{g} \mid f, g \in \mathbb{C}[x_0, \dots, x_n] \text{ homogeneous of the same degree, } g \notin P, f \in P\}}.$$

(3) For any $a = [a :_0:, \cdots : a_n] \in X$, the **tangent space** of X at a is

$$T_a X := Der(\mathcal{O}_{X,a}, a)$$

(4) $\dim X := \operatorname{tr.deg}_{\mathbb{C}} \mathbb{C}(X)$.

(5) $a \in X$ is smooth iff $\dim T_a X = \dim X$. $a \in X$ is singular if it is not smooth.

Remark 2.11. (Under the previous notations)

(1) If $a \in X \cap U_i$, then

$$\mathcal{O}_{X,a} \cong \mathcal{O}_{\phi_i(X \cap U_i), \phi_i(a)}$$

holds for $i = 0, \dots, n$.

Indeed. Take, WLOG, $i = 0$. Recall the map α, β in the proof of **Proposition** 2.2.2. We have seen that $phi_0(X \cap U_0) = V(\alpha(P))$. Note that if $g \in \mathbb{C}[x_0, \dots, x_n]$ is homogeneous and $g(a) \neq 0$, then $\alpha(g)(\phi_0(a)) \neq 0$. Hence α induces a homomorphism

$$\mathcal{O}_{X,a} \to \mathcal{O}_{\phi_i(X \cap U_i), \phi_i(a)}$$
$$\frac{f}{g} \mapsto \frac{\alpha(f)}{\alpha(g)}.$$

(surj.) Let $\frac{h}{k} \in \mathcal{O}_{\phi_i(X \cap U_i), \phi_i(a)}$, $\frac{h}{k} = \frac{\alpha(\beta(h) x_0^{\deg k})}{\alpha(\beta(k) x_0^{\deg h})}$.

(inj.). Clear.

Moreover, this map sends the ideal generated by P to the ideal generated by $P' = \alpha(P)$. Then the statement follows from

$$\mathcal{O}_{\phi_0(X\cap U_0),\phi_0(a)} \cong \frac{\mathcal{O}_{\mathbb{C}^n,\phi_0(a)}}{P'\mathcal{O}_{\mathbb{C}^n,\phi_0(a)}}.$$

See the proof of the theorem about the univity of Taylor series.

(2) If $X \subsetneqq H_i$ i.e. $X \cap U_i \neq \varnothing$, then $\mathbb{C}(X) \cong \mathbb{C}(\phi_i(X \cap U_i)$.

In particular, $\mathbb{C}(X) \cong \mathrm{Frac}(\mathcal{O}_{X,a})\ \forall a \in X$.

Indeed. The first isomorphism is induced as in (1). The second isomorphism follows the fact that

$$\mathbb{C}(\phi_i(X \cap U_i)) \cong \mathrm{Frac}(\mathcal{O}_{\phi_i(X\cap U_i),\phi_i(a)}).$$

Proposition 2.2.3. *Let $X = V(P) \subset \mathbb{P}^n$ be a projective variety. If $X \cap U_i \neq \varnothing$, $i = 0, \ldots, n$, and let $a \in X \cap U_i$. Then $\dim X = \dim \phi_i(X \cap U_i)$ and a is a smooth point of X iff $phi_i(a)$ is a smooth point of $\phi_i(X \cap U_i)$.*

Proof. From remark 2.11 (2), we have

$$\dim X = \mathrm{tr.deg}_{\mathbb{C}}\, \mathbb{C}(X) = \mathrm{tr.deg}_{\mathbb{C}}\, \mathbb{C}(\phi_i(X \cap U_i) = \dim \phi_i(X \cap U_i).$$

And from remark 2.11 (1), we have

$$Der_{\mathcal{O}_{X,a},a} \cong Der_{\mathcal{O}_{\phi_i(X\cap U_i),\phi_i(a)},\phi_i(a)} \cong Der_{R_{\phi_i(X\cap U_i)},\phi_i(a)}.$$

Moreover, $T_{\phi_i(a)} = Der_{R_{\phi_i(X\cap U_i)},\phi_i(a)} + \phi_i(a)$. Then

$$\dim T_a X \geq \dim X$$

and equality holds iff a is smooth in X. □

Remark 2.12. Let $X = V(P) \subset \mathbb{P}^n$ be a projective variety. Consider the affine cone over X

$$\mathcal{C} = \{a = (a_0, \ldots, a_n) \in \mathbb{C}^{n+1} \mid f(a) = 0,\ \forall f \in P\} \subset \mathbb{C}^{n+1}.$$

Then $\dim \mathcal{C} = \dim X + 1$

Example 2.2.2. *Let $f_1, \ldots, f_m \in \mathbb{C}[x_0, \ldots, x_n]$ be homogeneous of degree 1. Let $P := (f_1, \ldots, f_m)$ and $X := V(P)$. (Assume $X \neq \varnothing$). Then X is a projective variety of dimension $\dim X = n - \mathrm{rk}\, A$ where $A = (a_{jk}) \in \mathrm{Mat}_{m,n+1}(\mathbb{C})$ with $f_j = \sum_{k=0}^n a_{jk}x_k$.*

Indeed. *Let $i \in \{0, \ldots, n\}$ s.t. $a \in U_i$. Then $\phi_i(X \cap U_i) \subset \mathbb{C}^n$ is a linear space of dimension $n - \mathrm{rk}\, A$.*

Take, WLOG, $i = 0$. $X = \{[x_0 : \cdots : x_n] \mid \sum_{k=0}^n a_{jk}x_k = 0, j = 1, \ldots, m\}$. Take $y_k = \frac{x_k}{x_0}$, then

$$\phi_0(X \cap U_0) = \{(y_1, \ldots, y_n) \mid \sum_{k=1}^n a_{jk}y_k = -a_{j0}\}$$

Exercise 2.2.1. *In this case, X is homeomorphic to $\mathbb{P}^{\dim X}$.*

Exercise 2.2.2. *If $f = \sum_{i=0}^{n} c_i x_i$ is homogeneous of degree 1 and $H := V(f)$, then*

$$\mathbb{P}^n \setminus H \to \mathbb{C}^n$$

$$[a_0 : \cdots : a_n] \mapsto (\frac{a_0}{f_{a_0,\ldots,a_n}}, \ldots, \widehat{\frac{a_i}{f_{a_0,\ldots,a_n}}}, \ldots, \frac{a_0}{f_{a_0,\ldots,a_n}})$$

where $\frac{a_i}{f_{a_0,\ldots,a_n}}$ is such that $c_i \neq 0$. Such H is called an hyperplane.

Proposition 2.2.4. *Let $X \subset \mathbb{P}^n$ be a projective variety of $\dim X = n - 1$. Then $X = V(f)$ for some $f \in \mathbb{C}[x_0, \ldots, x_n]$ irreducible homogeneous polynomial.*

Such X are called hypersurfaces

Proof. Assume $X = V(P)$ for some homogeneous prime P. Let $f \in P \setminus 0$ be of minimum degree. Note that f is irreducible. Then $X = V(P) \subset V(f)$. Assume by contradiction that $\exists a \in V(f) \setminus X$.

Case 1. $\exists i \in \{0, \ldots, n\}$ s.t. $X \cap U_i \neq \varnothing$ and $a \in U_i$. Then

$$\phi_i(X \cap U_i) \subsetneqq \phi_i(V(f) \cap U_i) \subset \mathbb{C}^n$$

of the same dimension $n - 1$. Contradiction.

Case 2. No such i s.t. $X \cap U_i \neq \varnothing$. And $a \notin U_i$. Then $\exists j \in \{0, \ldots, n\}$ s.t. $X \subset H_j$. Hence $x_j \in P$. Since x_j is an element of minimal degree of P, $X = H_j$. By **Case 1.**, replace f with x_j. □

Example 2.2.3 (Quadrics)**.** *Let $f \in \mathbb{C}[x_0, \ldots, x_n]$ be homogeneous irreducible polynomial of* $\deg f = 2$. *WLOG, assume that*

$$f = \sum_{i=0}^{r} x_i^2$$

where r is called the rank of f. $X = V(f)$ is called a quadric in $\mathbb{P}^n$.

Consider $X \cap U_k$ for $k = 0, \ldots, n$.

Case 1. $k \leq r$. Then $\phi_k(X \cap U_k) = V(1 + \sum_{i \neq k} y_i^2) \subset \mathbb{C}^n$, $y_i = \frac{x_i}{x_k}$ is a smooth hypersurface since the Jacobian $(2y_0, \ldots, \widehat{2y_k}, \ldots, 2y_r)$ is full rank.

Case 2. $k > r$. Then $\phi_k(X \cap U_k) = V(\sum_{i=0}^{r} y_i^2)$, $y_i = \frac{x_i}{x_k}$. The Jacobian $(2y_0, \ldots, 2y_r)$ is 0 for any $(y_0, \ldots, \widehat{y_k}, \ldots, y_n) = (0, \ldots, 0, y_{r+1}, \ldots, \widehat{y_k}, \ldots, y_n)$. Hence If $r < n$, X contains singular points. X is smooth iff $r = n$.

Exercise 2.2.3. *$f = x_0^2 + \sum_{i=1}^{r} x_i^2$ is irreducible $\iff$ $\Delta = -4 \sum_{i=1}^{r} x_i^2 \in \mathbb{C}[x_1, \ldots, x_n]$ is not a square.*

2.3. **Divisors.**

Definition 2.3.1. Let $X = \mathbb{P}^n$ be a smooth projective variety. A prime divisor is a subvariety $Z \subset X$ of $\operatorname{codim} Z = 1$.

Definition 2.3.2. Let $\mathcal{Z}$ be the set of all prime divisors of X. We define the **group of divisors** $\operatorname{Div}(X)$ of X as the free abelian group generated by $\mathcal{Z}$.

$D \in \mathrm{Div}(X)$ is called a divisor of X and

$$D = \sum_{Z \in \mathcal{Z}} n_Z Z$$

where $n_Z \in \mathbb{Z}$ and only finitely many of $n_Z \neq 0$.

Remark 2.13. For any $Z \in \mathcal{Z}$ and $a \in Z$, if $a \in U_i$ for some i, then

$$\phi_i(Z \cap U_i) \subset \phi_i(X \cap U_i)$$

is a prime divisor.

Moveover, $\mathcal{O}_{X,a} \cong \mathcal{O}_{\phi_i(X \cap U_i), \phi_i(a)}$. Recall that we have defined the notion of the local equation of $\phi_i(Z \cap U_i)$ in $\phi_i(X \cap U_i)$, which is an irreducible $u_Z \in \mathcal{O}_{\phi_i(X \cap U_i), \phi_i(a)}$ viewed in $\mathcal{O}_{X,a}$.

Let $u \in \mathbb{C}(X) \setminus 0$. Then for any $Z \in \mathcal{Z}$, choose $a \in Z$ and a local equation $u_Z \in \mathcal{O}_{X,a}$.

Define the order of u at Z as $\mathrm{ord}_{\phi_i(Z \cap U_i)}(u)$ in affine case, where we identify $\mathbb{C}(X) = \mathbb{C}(\phi_i(X \cap U_i))$ and $a \in X \cap U_i$. One can prove that this definition does not depend on a.

Note that $\mathrm{ord}_Z(u) \neq 0$ only for finitely many Z. Then we can define $\mathrm{div}(u) := \sum_{Z \in \mathcal{Z}} \mathrm{ord}_Z(u) Z \in \mathrm{Div}(X)$. It is called a principal divisor.

Moreover, $\mathrm{div}(uv) = \mathrm{div}(u) + \mathrm{div}(v)$, $-\mathrm{div}(u) = \mathrm{div}(u^{-1})$. Hence the set $P(X)$ of principal divisors is a subgroup of $\mathrm{Div}(X)$.

Definition 2.3.3. The **divisor class group** of X is the group

$$\mathrm{Cl}(X) := \mathrm{Div}(X)/P(X).$$

Proposition 2.3.1. *$\mathrm{Cl}(\mathbb{P}^n) = \mathbb{Z}$.*

Proof. We have seen that $Z \in \mathbb{Z}$ corresponds to $Z = V(f)$ for some irreducible $f \in \mathbb{C}[x_0, \ldots, x_n]$. We can define $\deg(Z) := \deg(f)$. And this can be extended to a group homomorphism

$$\begin{aligned} \deg : \mathrm{Div}(\mathbb{P}^n) &\to \mathbb{Z} \\ \sum_{Z \in \mathcal{Z}} n_Z Z &\mapsto \sum_{Z \in \mathcal{Z}} n_Z \deg(Z). \end{aligned}$$

We have the following short exact sequence

$$0 \to \ker(\deg) \to \mathrm{Div}(\mathbb{P}^n) \xrightarrow{\deg} \mathbb{Z} \to 0.$$

Claim. $\ker(\deg) = P(\mathbb{P}^n)$.

Indeed. Let $u = \frac{f}{g} \in \mathbb{C}(\mathbb{P}^n)$ where $f, g \in \mathbb{C}[x_0, \ldots, x_n]$ are homogeneous polynomials of the same degree. Factorize f, g

$$f = \prod_{k=1}^{M} f_k^{\mu_k}, \ \deg(f) = \sum \mu_k \deg(f_k)$$

$$f = \prod_{k=1}^{N} g_k^{\nu_k}, \ \deg(g) = \sum \nu_k \deg(g_k).$$

Hence $\operatorname{div}(u) = \sum_{k=1}^{M} \mu_k V(f_k) - \sum_{k=1}^{N} \nu_k V(g_k)$ and $\deg(u) = \deg(f) - \deg(g) = 0$.

On the other hand, if $D \in \operatorname{Div}(\mathbb{P}^n)$ has $\deg(D) = 0$, we can write

$$D = \sum_{k=1}^{M} \mu_k Z_k - \sum_{l=1}^{N} \nu_l Y_l$$

where μ_k, ν_l are positive and $Z_k = V(f_k), Y_l = V(g_l)$ for some f_k, g_l.

Therefore $D = \operatorname{div}(\frac{\prod f_k^{\mu_k}}{\prod g_l^{\nu_l}})$. □

Remark 2.14. Let $X \subset \mathbb{P}^n$ be a smooth projective variety of $\dim X = 1$. We have the following exact sequence

$$0 \to \mathcal{J}(X)(\cong \mathbb{C}^g/\Lambda) \to \operatorname{Pic}(X) \xrightarrow{\deg} \mathbb{Z} \to 0$$

where $\Lambda(\cong \mathbb{Z}^{2g}) \leq \mathbb{C}^g$ is a lattice of maximal rank and g is the topological genus of X.

2.4. **Grassmannians.** A good reference is [2]

Definition 2.4.1. Let V be a n-dimensional vector space over $\mathbb{C}$. Let $1 \leq k \leq n$. The grassmannian is the set

$$Gr(k, V) := \{W \subset V \mid W \text{ is a subspace of } \dim(W) = k\}.$$

We write $Gr(k, n)$ for $V = \mathbb{C}^n$.

Remark 2.15. This definition can be extended to any field $\Bbbk$ of $\operatorname{char} \Bbbk \neq 2$.

Example 2.4.1. (1) $Gr(1, n) = \mathbb{P}^{n-1}$,
(2) $Gr(n-1, n) = (\mathbb{P}^{n-1})^\vee$.

In general, given $W \in Gr(k, n)$ and choose a basis of W: $v_i = \sum_j a_{ij} e_j,\ i = 1, \ldots, k$ where e_i is the standard basis of $\mathbb{C}^n$. We can represent W with $A = (a_{ij}) \in \operatorname{Mat}_{k,n}(\mathbb{C})$ ($\operatorname{rk} A = k$).

For $A, A' \in \operatorname{Mat}_{k,n}(\mathbb{C})$, if there exists $C \in \operatorname{GL}_k(\mathbb{C})$ such that $A' = CA$, then they represent the same subspace W.

Now for any multi-index $I = (i_1 < \cdots < i_k)$, define

$$U_I := \{W \in Gr_{k,n} \mid W \cap \operatorname{Span}(e_j \mid j \notin I) = \{0\}\}.$$

Note that

$$\begin{aligned} W \in U_I &\iff W \oplus \operatorname{Span}(e_j \mid j \notin I) = \mathbb{C}^n \\ &\iff \text{if } v_1, \ldots, v_k \text{ form a basis of } W \text{ then } v_1, \ldots, v_k, e_j,\ j \notin I \text{ form a basis of } \mathbb{C}^n \\ &\iff \text{if } A \in \operatorname{Mat}_{k,n}(\mathbb{C}) \text{ represent } W \text{ then } \det \tilde{A} \neq 0 \\ &\iff \det(A^{(I)} := (A^{i_1}, \ldots, A^{i_k}) \neq 0. \end{aligned}$$

It follows that any $W \in U_I$ is represented by a unique $A \in \operatorname{Mat}_{k,n}(\mathbb{C})$ s.t. $A^{(I)} = I_K$. Indeed, if W is represented by A, then $A^{(I)} \in \operatorname{GL}_k(\mathbb{C})$, hence $A' := (A^{(I)})^{-1} A$ represents W and $(A')^{(I)} = I_k$.

Define

$$\begin{aligned}\phi_I : U_I &\to \mathbb{C}^{k(n-k)} (\cong \mathrm{Mat}_{k,n-k}(\mathbb{C}))\\ W &\mapsto (A^{(j)} \mid j \notin I).\end{aligned}$$

where A is the representative of W s.t. $A^{(I)} = I_k$. ϕ_I is a bijection. And we can endow U_I with the unique topology such that ϕ_I is homeomorphism w.r.t the classical topology on $\mathbb{C}^{k(n-k)}$.

Since $Gr(k,n) = \cup_I U_I$, we can consider the smallest topology on $Gr(k,n)$ generated by the topologies on U_I. And with this topology $Gr(k,n)$ is a compact, Hausdorff topological space. And $\phi_J \circ \phi_I^{-1}$ is $\mathcal{C}^\infty$. We have that $Gr(k,n)$ is a smooth manifold.

Remark 2.16. There is a better way to proceed. Consider the action of $\mathrm{GL}_k(\mathbb{C})$

$$\begin{aligned}\mathrm{GL}_k(\mathbb{C}) \times \{A \in \mathrm{Mat}_{k,n}(\mathbb{C}) \mid \operatorname{rk} A = k\} &\to \{A \in \mathrm{Mat}_{k,n}(\mathbb{C}) \mid \operatorname{rk} A = k\}\\ (C, A) &\mapsto CA.\end{aligned}$$

Then $Gr(k,n) = \{A \in \mathrm{Mat}_{k,n}(\mathbb{C}) \mid \operatorname{rk} A = k\} / \mathrm{GL}_k(\mathbb{C})$ with the quotient topology and it is a smooth manifold.

Lemma 2.4.1. *Let $\tau \in \bigwedge^k V \setminus \{0\}$ where V is a n-dimensional vector space over $\mathbb{C}$.*

Then $\dim\{w \in W \mid \tau \wedge w = 0\} \le k$ and the equality holds iff τ is decomposable, i.e. $\tau = v_1 \wedge \cdots \wedge v_k$.

Proof. Let $w_1, \ldots, w_m$ be the basis of $\{w \in W \mid \tau \wedge w = 0\}$. Complete $w_1, \ldots, w_m$ to a basis of V as $w_1, \ldots, w_m, w_{m+1}, \ldots, w_n$. Then $\tau = \sum_I P_I w_I$ for multi-index $I = (i_1 < \cdots < i_k)$. We have

$$\begin{aligned}0 = \tau \wedge w_j &= \sum_I P_I w_I \wedge w_j\\ &= \sum_{\substack{I\\ j \notin I}} P_I w_I \wedge w_j\\ &\Rightarrow P_I = 0 \text{ if } j \notin I,\ \forall j = 1, \ldots, m.\end{aligned}$$

But $|I| = k$ and $\tau \ne 0$, hence $k \ge m$. Moreover, if $k = m$, then $P_I = 0$ if $I \ne \{1, \ldots, m\}$. Hence $\tau = w_1 \wedge \cdots \wedge w_k$.

Conversely, if $\tau = v_1 \wedge \cdots \wedge v_k \ne 0$, then

$$\dim\{w \in W \mid \tau \wedge w = 0\} = \mathrm{Span}(v_1, \ldots, v_k)$$

and it has dimension k. □

Theorem 2.4.1 (Plüker embedding)**.** *We have an injective map.*

$$P : Gr(k,n) \to \mathbb{P}^{\binom{n}{k}-1}$$

Moreover, the image of P, still denoted by $Gr(k,n)$, is a closed algebraic set.

Proof. Let $W \in Gr(k,n)$. Choose a basis $\{v_1,\dots,v_k\}$ of W and consider $v_1 \wedge \cdots \wedge v_k \in \bigwedge^k \mathbb{C}^n$. Define the map

$$\begin{aligned} P : Gr(k,n) &\to \mathbb{P}(\bigwedge^k \mathbb{C}^n) \\ W &\mapsto [v_1 \wedge \cdots \wedge v_k]. \end{aligned}$$

Identifying $\bigwedge^k \mathbb{C}^n \cong \mathbb{C}^{\binom{n}{k}}$ we get map to the projective space.

Moreover, P is injective. Indeed, if $P(W) = P(W')$, then for basis $\{v_i\}$ for W and $\{v_i'\}$ for W',

$$v_1 \wedge \cdots \wedge v_k = \lambda v_1' \wedge \cdots \wedge v_k' \ \lambda \in \mathbb{C}^\times.$$

Since $W = \{v \in \mathbb{C}^n \mid v_1 \wedge \cdots \wedge v_k \wedge v = 0\} = W'$. And moreover,

$$\begin{aligned} P(W) &= [v_1 \wedge v_k] \\ &= [\sum_{I=(1\le i_1<\cdots<i_k\le n)} P_I e_I] \\ &= [P_I \mid I]. \end{aligned}$$

P_I are called the Plücker coordinates of W.

Claim. $Gr(k,n) \subset \mathbb{P}^{\binom{n}{k}-1}$ is a closed algebraic set.

Indeed. We know that $Gr(k,n) = \{[\tau] \in \mathbb{P}(\bigwedge^k \mathbb{C}^n) \mid \text{s.t. } \tau = v_1 \wedge \cdots \wedge v_k\}$. Consider the linear map

$$\begin{aligned} f : \mathbb{C}^n &\to \bigwedge^{k+1} \mathbb{C}^n \\ v &\mapsto \tau \wedge v. \end{aligned}$$

τ decomposable $\iff$ $\dim\ker f = k$ $\iff$ $\operatorname{rk} f = n-k$.

Let $\mathcal{B} := \{e_1,\dots,e_n\}$ be the standard basis of $\mathbb{C}^n$ and $\mathcal{C} := \{e_I \mid I = (i_1 < \cdots < i_{k+1})\}$ be the basis of $\bigwedge^{k+1} \mathbb{C}^n$. Represent f w.r.t. these basis as

$$M^{\mathcal{B}}_{\mathcal{C}}(f) = A \in \operatorname{Mat}_{\binom{n}{k+1},n}(\mathbb{C})$$

where entries of A are coefficient of τ. Since τ is decomposable, the determinants of all $(n-k+1)$ minors of A are 0, which are polynomials in P_I. $\square$

Example 2.4.2. *The first non-trivial example is $Gr(2,4) \subset \mathbb{P}^5$.*

$U_I \cong \mathbb{C}^4$, hence $Gr(2,4)$ is an hypersurface in $\mathbb{P}^5$.

Theorem 2.4.2. *$Gr_{k,n} \subset \mathbb{P}^{\binom{n}{k}-1}$ is a projective variety of dimension $k(n-k)$. Moreover, the ideal generated by*

$$\sum_{a=1}^{k+1} (-1)^a P_{i_1,\dots,i_k,j_a} P_{j_1,\dots,\widehat{j_a},\dots,j_{k+1}}$$

for any two sequence $1 \le i_1 < \cdots < j_{k-1} \le n$, $1 \le j_1 < \cdots < j_{k+1} \le n$ is prime and $V(P) = Gr_{k,n}$.

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