Ref Lie groups and Lie algebras. Etingof
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An Introduction to Lie Groups and Lie algebras A. Kirillov, Jr.

GTM 9 Humphreys.
Bourbaki 4-6.

Lie 代数 邓少奇 等

Char $k = 0$ $k = \bar{k}$ $=$ $\mathbb{C}$

线性代数. 群. 环. 域

Part I.

- Lie algebras and examples
- Universal eveloping algebras. PBW Thm. relation between Lie group and Lie algebra
- sl_2 的表示. "BB-localization of sl_2". $U_q(sl_2)$ quantum groups.
- Solvable and nilpotent Lie algebra.
- Semisimple Lie algebra and its structures
- root systems Weyl groups and Dynkin diagrams quiver and representation of associative algebra
- Classification of semisimple Lie algebra.

Part II.

- Representation theory of semisimple Lie algebra.
- Weyl character Formula.

Sophus Lie. 1870s
Killing, Engel. 1880s
Cartan 1894
1914

Lecture 1. $k = \mathbb{C}$

§1.1 Basic concepts and examples

Def 1.1.1 A Lie algebra over $\mathbb{C}$ is a vector space $\mathfrak{g}/\mathbb{C}$, equipped with an operation (Lie bracket)

$$[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying the following

(1) $[-,-]$ is bilinear.

(2) $[x,x] = 0 \quad \forall x \in \mathfrak{g}$

(3) (Jacobi identity) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$

Remark (Exer) (1) + (2) $\Rightarrow$ $\boxed{(2') \quad [x,y] = -[y,x]}$

(2') $\Leftrightarrow$ (2) if char(k) $\neq 2$.

Def 1.1.2 A Lie algebra $\mathfrak{g}$ is called abelian if

$$[x,y] = 0 \quad \forall x, y \in \mathfrak{g}$$

Def 1.1.3 $\mathfrak{g}, \mathfrak{g}'$ = Lie algebras. A homomorphism (resp. isomorphism) of Lie algebras is hom (resp. iso) of vector spaces

$$\varphi : \mathfrak{g} \to \mathfrak{g}'$$

s.t.

$$\varphi([x,y]) = [\varphi(x), \varphi(y)] \quad \forall x, y \in \mathfrak{g}.$$

Def. 1.1.4 $\mathfrak{g}$ = Lie algebra $\mathfrak{h} \subset \mathfrak{g}$.

$\mathfrak{h}$ is called a Lie subalgebra if it is closed under $[-,-]$

(Lie) ideal if moreover, $\forall x \in \mathfrak{g}, y \in \mathfrak{h}$ $[x,y] \in \mathfrak{h}$.

$(A/\mathbb{B},\ +,\ \cdot) +$ 数乘.

Example 1.1.5. $A =$ associative algebra (e.g. $\mathrm{Mat}_n(\mathbb{C})$)

$$[x, y] = xy - yx$$

Example 1.1.6 A derivation of A is a linear map.

$$d : A \longrightarrow A$$

s.t. $\forall\, a, b \in A$.

$$d(ab) = d(a)b + a\,d(b) \qquad \text{(Leibniz rule)}$$

$\mathrm{Der}(A) := \{\text{all derivation of } A\} \subset \mathrm{End}(A)$

For Lie algebra $\mathfrak{g}$, $d([a,b]) = [d(a), b] + [a, d(b)]$

$\mathrm{Der}(\mathfrak{g}) \subset \mathrm{End}(\mathfrak{g})$ is a Lie subalgebra.

$$\mathrm{ad}_x : \mathfrak{g} \longrightarrow \mathfrak{g}, \quad \forall x \in \mathfrak{g}, \qquad \text{(inner derivation)}$$
$$y \longmapsto [x, y]$$

Example 1.1.7. $V =$ n-dim vec space$/\mathbb{C}$

$\mathfrak{gl}(V) := \mathrm{End}(V)$ $\hookrightarrow$ $\mathfrak{gl}_n = \mathrm{Mat}_n(\mathbb{C})$

$\cup$ $\qquad\qquad\qquad \cup$

$\mathfrak{sl}(V) := \{x \in \mathfrak{gl}(V) \mid tr(x) = 0\}$ $\qquad$ $\mathfrak{sl}_n = \{A \in \mathfrak{gl}_n \mid tr(A) = 0\}$

$$\mathfrak{g} := \{x \in \mathfrak{gl}_n \mid Mx + x^T M = 0\}$$

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_\ell \\ 0 & I_\ell & 0 \end{pmatrix} \qquad \mathfrak{g} = \mathfrak{so}_{2\ell+1} \subset \mathfrak{gl}_{2\ell+1}$$

$$M = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix} \qquad \mathfrak{g} = \mathfrak{sp}_\ell \subset \mathfrak{gl}_{2\ell}.$$

$$M = \begin{pmatrix} 0 & I_\ell \\ I_\ell & 0 \end{pmatrix} \qquad \mathfrak{g} = \mathfrak{so}_{2\ell} \subset \mathfrak{gl}_{2\ell}.$$

(is linear)

Remark. (Ado's thm) Any fin-dim Lie algebra is a Lie subalgebra of $\mathfrak{gl}_n$.

Example. 1.1.8 V = n-dim vec space V.

A (complete) flag in V

$\mathcal{F} = (0 = V_0 \subset V_1 \subset \cdots \subset V_n = V)$

$\dim V_i = i$

Let

$\mathfrak{t} = \{X \in \mathfrak{gl}(V) \mid X(V_i) \subset V_i,\ 0 \le i \le n\} \subset \mathfrak{gl}(V)$

$\cup$

$\mathfrak{n} = \{X \in \mathfrak{gl}(V) \mid X(V_i) \subset V_{i-1}\ \ 0 \le i \le n\}$

Ref. Finite group: An introduction. J-P. Serre

Grp $\longrightarrow$ LieAlg.

Lemma. 1.1.9 $\mathfrak{g}$ = Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ ideal.

Then

(1) $\mathfrak{g}/\mathfrak{h}$ has a natural Lie algebra structure.

(2) $\phi: \mathfrak{g}_1 \to \mathfrak{g}_2$ = a hom. of Lie algebras then

(2.1) $\ker \phi \subset \mathfrak{g}_1$ is an ideal in $\mathfrak{g}_1$

(2.2) $\operatorname{im} \phi \subset \mathfrak{g}_2$ is an Lie subalgebra in $\mathfrak{g}_2$

(2.3) $\mathfrak{g}_1/\ker\phi \cong \operatorname{im}\phi$

Sketch of proof. (1) $[X+\mathfrak{h}, y+\mathfrak{h}] := [x, y] + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$

(2) (2.1) $\forall x \in \mathfrak{g}\quad y \in \ker\phi$

$\phi([x, y]) = [\phi(x), \phi(y)] = [\phi(x), 0] = 0 \Rightarrow [x, y] \in \ker\phi$

(2.2). (2.3) $\square$

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$[I_1, I_2] = \mathrm{span}_{\mathbb{C}} \{ \underline{[x, y]} \mid x \in I_1, y \in I_2 \}$

Lemma 1.1.10 $I_1, I_2 \subset \mathfrak{g}$. ideals Then $I_1 \cap I_2$, $I_1 + I_2$, $[I_1, I_2]$ are ideals.

proof $\forall x \in \mathfrak{g}$. $y_1 \in I_1 \cap I_2$.

$[x, y_1] \in I_1$, $\in I_2$ $\Rightarrow$ $[x, y_1] \in I_1 \cap I_2$.

$I_1 + I_2 \ni y_2 = \underset{I_1}{z_1} + \underset{I_2}{z_2}$.

$[x, y_2] = [x, z_1 + z_2] = \underset{I_1}{[x, z_1]} + \underset{I_2}{[x, z_2]} \in I_1 + I_2$

$[I_1, I_2] \ni y_3 = [t_1, t_2]$

$[x, y_3] = [x, [t_1, t_2]] = \underset{[I_1, I_2]}{[t_1, [x, t_2]]} + \underset{[I_1, I_2]}{[[x, t_1], t_2]}$

$\in [I_1, I_2]$ □

Def 1.1.11 $[\mathfrak{g}, \mathfrak{g}]$ is called the commutant of $\mathfrak{g}$.

Lemma 1.1.12. (1) $\mathfrak{g} / [\mathfrak{g}, \mathfrak{g}]$ is abelian.

(2) $\forall I \subset \mathfrak{g}$, if $\mathfrak{g}/I$ is abelian, the $I \supset [\mathfrak{g}, \mathfrak{g}]$

proof. (1) $\forall x, y \in \mathfrak{g}$, $[x, y] \in [\mathfrak{g}, \mathfrak{g}]$

(2) $\forall x, y \in \mathfrak{g}$, $[x, y] \in I$. $\Rightarrow$ $[\mathfrak{g}, \mathfrak{g}] \subset I$. □

$\mathfrak{g}$ $\{x_i\}$ $[x_i, x_j] = \sum_k c_{ij}^k x_k$.

$c_{ij}^k \in \mathbb{C}$ structure constant.

§1.2. Representation of Lie algebras

Def. 1.2.1 $\mathfrak{g}$ = Lie algebra $/\mathbb{C}$

A representation of $\mathfrak{g}$ (or a $\mathfrak{g}$-module) is a vector space $V/\mathbb{C}$ equipped with a homomorphism of Lie algebras

$$\rho = \rho_V : \mathfrak{g} \to \mathfrak{gl}(V)$$

A morphism (resp. isomorphism) of representation V, and W.
(intertwining operator)

is a linear map (resp. isomorphism)

$$A : V \to W$$

which commutes with the $\mathfrak{g}$-action.

$$A\rho_V(b) = \rho_W(b) A \qquad \forall\, b \in \mathfrak{g}.$$

Example 1.2.2.

(1) (trivial rep) $\quad \rho(x) = 0 \in \mathfrak{gl}(V) \qquad \forall\, x \in \mathfrak{g}$

(2) (adjoint rep) $\quad \rho(x) = \mathrm{ad}_x \qquad \forall\, x \in \mathfrak{g}.$

Def. 1.2.3 A subrep. of V is a $\mathfrak{g}$-invariant subspace $W \subset V$.

i.e., $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, $x \mapsto \rho_x$ $\qquad \forall\, x \in \mathfrak{g}$, $w \in W$. $\quad \rho_x(w) \in W$.

Example 1.2.4.

(1) (quotient) $\quad V$ = rep of $\mathfrak{g}$, $W \subset V$ sub rep.

Then V/W is a rep of $\mathfrak{g}$.

(2) (direct sum) $\quad V, W$ = rep of $\mathfrak{g}$.

$$\rho_{V\oplus W}(x) = \rho_V(x) \oplus \rho_W(x) \qquad \forall\, x \in \mathfrak{g}.$$

$(V \oplus W, \rho_{V\oplus W})$ is a rep of $\mathfrak{g}$.

(3) (tensor product) $\rho_{V\otimes W}(x) = \rho_V(x) \otimes 1_W + 1_V \otimes \rho_W(x) \quad \forall x \in \mathfrak{g}$

(4) (dual rep) V = rep of $\mathfrak{g}$. $\quad V^*$ = dual of V

$$\rho_{V^*} : \mathfrak{g} \longrightarrow \mathfrak{gl}(V^*)$$
$$x \longmapsto \left[\rho_{V^*}(x) = -\rho_V(x)^* : \begin{array}{l} V^* \longrightarrow V^* \\ f \longmapsto -f\circ\rho_V(x) \end{array}\right]$$

<u>check</u> $\rho_{V^*}([x,y])(f) = -f\circ\rho_V([x,y])$

$$= -f\circ[\rho_V(x),\rho_V(y)]$$

$$= -f\circ(\rho_V(x)\rho_V(y) - \rho_V(y)\rho_V(x))$$

$$= \underline{-f\circ\rho_V(x)}\,\rho_V(y) + f\,\rho_V(y)\rho_V(x)$$

$$= (\rho_{V^*}(x)\circ f)\circ\rho_V(y) - (\rho_{V^*}(y)\circ f)\circ\rho_V(x)$$

$$= -\rho_{V^*}(y)\,\rho_{V^*}(x)\circ f + \rho_{V^*}(x)\,\rho_{V^*}(y)\circ f$$

$$= [\rho_{V^*}(x), \rho_{V^*}(y)]\circ f$$

$S^n V$, $\Lambda^n V$.

Def 1.2.5 V = rep of $\mathfrak{g}$.

$$V^{\mathfrak{g}} = \{ v \in V \mid \rho_x(v) = 0 \quad \forall x \in \mathfrak{g} \}$$

Def 1.2.6 A rep V of $\mathfrak{g}$ is called

(1) irreducible if it has no non-trivial subrep.

i.e. $\forall W \subset V \quad W = 0$ or V.

(2) indecomposable if $\forall\ V \cong V_1 \oplus V_2$ we have either $V_1 = 0$ or $V_2 = 0$

(3) completely reducible if $V \cong (\oplus$ irrd. reps$)$

Main problem.

(1) Classify all irrd. rep

(2) $V = (\bigoplus \underline{\text{irrd. reps}})$

↑ Find.

(3) For which "objects" are all rep completely reducible.

↑ semi simple Lie algebra.

Remark Any fin-dim rep $V \cong \bigoplus$ (indecom. reps) uniquely up to the order of summands.

Krull-Schmidt theorem (Module theory).

Lemma 1.2.7 (Schur's Lemma) V, W = irrd. fin-dim reps of $\mathfrak{g}$. then

(1) $\mathrm{Hom}_{\mathfrak{g}}(V, W) = 0$ if $V \not\cong W$.

(2) $\mathrm{End}_{\mathfrak{g}}(V) = \mathbb{C}$

proof (1) Suppose $V \not\cong W$ and $\mathrm{Hom}_{\mathfrak{g}}(V, W) \neq 0$

Let $A \in \mathrm{Hom}_{\mathfrak{g}}(V, W)$ be a nonzero morphism. then.

$\mathrm{im}(A) \subset W$ is a nonzero subrep, hence $\mathrm{im}(A) = W$

Then $\ker(A) \subsetneq V$ is a proper subrep $\Rightarrow$ $\ker(A) = 0$

A is an isomorphism $\times$

(2) $A \in \mathrm{End}_{\mathfrak{g}}(V)$, Let $\lambda \in \mathbb{C}$ is an eigenvalue of A.

Then $A - \lambda \cdot \mathrm{id} \in \mathrm{End}_{\mathfrak{g}}(V)$ but not a isomorphism $\overset{(1)}{\Rightarrow}$ $A - \lambda \mathrm{id} = 0$

$\Rightarrow$ $A = \lambda \cdot \mathrm{id}$. acting by scalar. □

Remark. (2) is not true when $k = \overline{k}$

Cor 1.2.8. The center of $\mathfrak{g}$ acts on irrd. rep V by scalar.
In particular, if $\mathfrak{g}$ is abelian, then every irrd. rep of $\mathfrak{g}$ is 1-dim

Proof $\quad \mathfrak{z}(\mathfrak{g}) = \{ x \in \mathfrak{g} \mid [x,y]=0 \quad \forall y \in \mathfrak{g} \}$.

For $x \in \mathfrak{z}(\mathfrak{g})$

$\rho_V(x) : V \to V$

$\forall y \in \mathfrak{g}$,

$$\mathfrak{gl}(V) \ni \rho_V(x)\rho_V(y) - \rho_V(y)\rho_V(x) = [\rho_V(x), \rho_V(y)] = \rho_V([x,y]) = 0$$

$$\rho_V(x)\rho_V(y) = \rho_V(y)\rho_V(x)$$

$\Rightarrow \rho_V(x) \in \mathrm{End}_{\mathfrak{g}}(V) \Rightarrow x$ acts by scalar.

If $\mathfrak{g}$ is abelian, $\forall x \in \mathfrak{g}$, $\rho_V(x) = \lambda_x \cdot id$, $\quad \lambda_x \in \mathbb{C}$

So any $v \underset{\neq 0}{\in} V$ spans an 1-dim subspace of V

V irrd. $\Rightarrow$ 1-dim subspace $= V \Rightarrow \dim V = 1$ $\quad \square$

Cor. 1.2.9. (V_i) = irrd reps.

$V = \bigoplus_i n_i V_i = \bigoplus_i V_i^{\oplus n_i}$ $\qquad$ (Fulton, ... Rep theory, a frist course)

$W = \bigoplus_i m_i V_i = \bigoplus_i V_i^{\oplus m_i}$

complete reps $/\mathbb{C}$ of $\mathfrak{g}$.

Then we have a natural linear isomorphism

$$\mathrm{Hom}_{\mathfrak{g}}(V,W) \cong \bigoplus \mathrm{Mat}_{m_i \times n_i}(\mathbb{C})$$

Moreover $V \cong W$ is an isomorphism of asso. algebra