Lecture 2.

§2.0 Review

Let R be a commutative ring

Def 2.0.1 (1) An R-module is an abelian group M. with a map

$$R \times M \to M$$
$$(a, m) \mapsto am.$$

that satisfies the following

① Associativity $(ab)m = a(bm)$

② Distributivity $(a+b)m = am + bm$ $\forall a, b \in R$

$a(m+m') = am + am'$ $m, m' \in M$

③ Unit $1m = m$.

(2) M, N = R-modules A homomorphism (R-linear map) is a group homomorphism

$$\varphi: M \to N$$

s.t. $\varphi(am) = a\varphi(m)$ $\forall a \in R$ $m \in M$.

Remark $\mathbb{Z}$-module = abelian group

k-module = k-vector space.

Def 2.0.2 An (associative) R-algebra is a ring A equipped with an R-module structure s.t.

$$r(mm') = (rm)m' = (mr)m' = m(rm') \quad \forall m, m' \in A, r \in R$$

Recall some (multi-) linear algebra

V/k

The tensor algebra of V is the $\mathbb{Z}_{\geq 0}$-graded associative algebra

$$TV := \bigoplus_{n \geq 0} V^{\otimes n} \qquad (\deg V^{\otimes n} = n)$$

with the multiplication given by $a \cdot b := a \otimes b$ for $a \in V^{\otimes m}$, $b \in V^{\otimes n}$.

Here by a $\mathbb{Z}$-graded asso. algebra we mean an asso. algebra A that its underlying abelian group is endowed with a decomposition $A = \bigoplus_{k \in \mathbb{Z}} A_k$

s.t. $A_l \cdot A_m \subset A_{l+m}$ $\quad \forall\, l, m \in \mathbb{Z}$.

Universal property. A = k-algebra. $f: V \to A$ linear map

$$\begin{array}{ccc} V & \xrightarrow{f} & A \\ {\scriptstyle i}\downarrow & \circlearrowleft & \nearrow {\scriptstyle \exists!\, \tilde{f}} \\ TV & & \end{array}$$

Consider the (graded) two-side ideal of TV

$$J := \langle x \otimes y - y \otimes x \mid x, y \in V \rangle$$

The quotient $TV/J =: S(V)$ is called the symmetric algebra on V.

Universal property.

$$\begin{array}{ccc} V & \xrightarrow{f} & A \\ {\scriptstyle i}\downarrow & \circlearrowleft & \nearrow {\scriptstyle \exists!\, \tilde{f}} \\ S(V) & & \end{array}$$

given by $\tilde{f}(v_1 \cdots v_d) = f(v_1) \cdots f(v_d)$

§2.1 Universal enveloping algebra.

Def 2.1.1. The UEA of $\mathfrak{g}$ is

$$U(\mathfrak{g}) := T\mathfrak{g}/I$$

where I is the ideal generated by

$$xy - yx - [x,y] \qquad \forall x,y \in \mathfrak{g}.$$

Prop 2.1.2

(1) Let $I' \subset T\mathfrak{g}$ be an ideal and $\rho: \mathfrak{g} \to T\mathfrak{g}/I'$ the natural linear map. Then ρ is a homomorphism of Lie algebras $\Leftrightarrow$ $I' \supset I$. so that $T\mathfrak{g}/I'$ is a quotient of $T\mathfrak{g}/I = U(\mathfrak{g})$

(2) Let A be any asso. k-algebra. Then the map

$$\mathrm{Hom}_{Asso}(U(\mathfrak{g}), A) \longrightarrow \mathrm{Hom}_{Lie}(\mathfrak{g}, A)$$

$$U(\mathfrak{g}) \xrightarrow{\phi} A \quad \longmapsto \quad \mathfrak{g} \xrightarrow{\rho} U(\mathfrak{g}) \xrightarrow{\phi} A, \quad \phi\circ\rho$$

is a bijection.

proof (1) ρ is a hom of Lie algebra

$\Leftrightarrow$ $\rho(x)\rho(y) - \rho(y)\rho(x) - \rho([x,y]) = 0 \in I' \qquad \forall x,y \in \mathfrak{g}.$

$\Leftrightarrow$ $I \subset I'$

(2) Given a Lie algebra homomorphism $f: \mathfrak{g} \to A$.

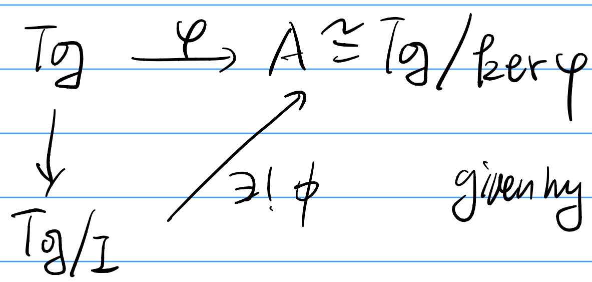
$\exists! \varphi$,

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{f} & A \\ \downarrow & \circlearrowleft \nearrow & \\ T\mathfrak{g} & \exists! \varphi & \end{array}$$

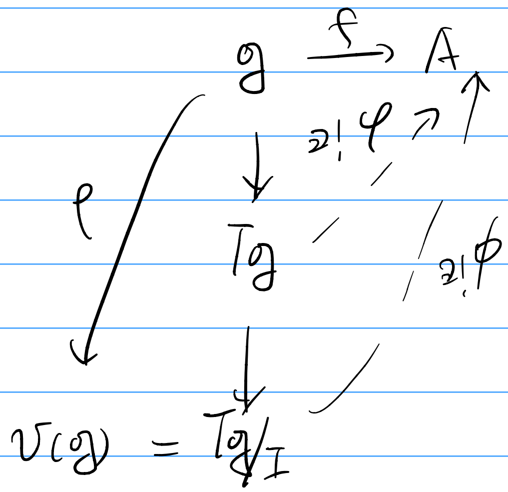
given by $\varphi(x) := f(x) \qquad \forall x \in \mathfrak{g}.$

$$\varphi(xy-yx) = \varphi(x)\varphi(y) - \varphi(y)\varphi(x)$$
$$= f(x)f(y) - f(y)f(x)$$
$$= [f(x), f(y)]$$
$$= f([x,y])$$
$$= \varphi([x,y])$$

$I \subset \ker\varphi$ $\quad \exists!\, \phi$

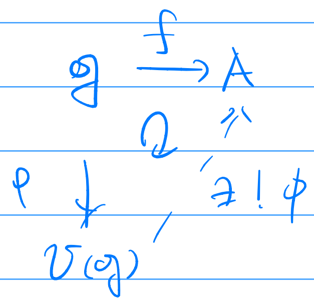


given by $\phi(x) = \varphi(x)$



$\phi \leftrightarrow f = \phi \circ \rho$

$\Rightarrow$ universal property of $U(\mathfrak{g})$



Claim $U(\mathfrak{g})$ is not graded

Indeed. If we try to define $\deg(x_1 \cdots x_k) = k$ $x_i \in \mathfrak{g}$.

$\deg(xy) = \deg(yx) = 2$

$\deg(xy - yx) = \deg([x, y]) = 1$

Instead, we have a "weaker" structure: filtration.

Recall

◇ A $\mathbb{Z}_{\geq 0}$ - filtration algebra is an algebra A equipped with a filtration

$$0 = F_{-1}A \subset F_0 A \subset F_1 A \subset \cdots \subset F_n A \subset \cdots$$

such $1 \in F_0 A$

$\bigcup_{n \geq 0} F_n A = A$ $\quad\quad F_i A \cdot F_j A \subset F_{i+j} A$.

In particular, if A is generated by $\{x_i\}$, then a filtration on A can be obtained by declaring x_i to be degree 1.

$$F_n A = (F_1 A)^n = \text{Span of all words in } x_i \text{ of deg} \leq n$$

"$x_1 x_2 \cdots x_{n-1}$"

◇ If $A = \bigoplus_{i \geq 0} A_i$ is $\mathbb{Z}_{\geq 0}$ - graded then we can define a filtration on A by setting

$$F_n A = \bigoplus_{i=0}^{n} A_i$$

For a filtered algebra, we can define its associated graded algebra

$$\operatorname{gr} A := \bigoplus_{n \geq 0} \operatorname{gr}_n A$$

where $\operatorname{gr}_n A := F_n A / F_{n-1} A$

Multiplication.

$\forall a \in \operatorname{gr}_i A,\ b \in \operatorname{gr}_j A$. we can take the representative elements

$F_i A / F_{i-1} A$ $\quad\quad F_j A / F_{j-1} A$

$F_{i+j}A \twoheadrightarrow F_{i+j}A / F_{i+j-1}A$

$\tilde{a}\tilde{b} \longmapsto ab$

$\tilde{a} \in F_iA$ $\tilde{b} \in F_jA$, then ab is the image of $\tilde{a}\tilde{b}$ in $gr_{i+j}A$

Prop 2.1.3. If $gr\,A$ is a domain (has no zero divisors), then so is A

proof Suppose that A is not a domain.

$A = 0$ trivial.

$A \neq 0$. then $\exists \overset{\neq 0}{a}, \overset{\neq 0}{b} \in A$ s.t. $ab = 0$

$\exists\, i,j \geq 0$ $a \in F_iA$. $b \in F_jA$ $a \notin F_{i-1}A$ $b \notin F_{j-1}A$.

then the image $[a]_i \in gr_iA$ $[b]_j \in gr_jA$ are nonzero

with $[a]_i[b]_j = [ab]_{i+j} = 0$ $\times$ $\square$

Now we define the filtration on $U(\mathfrak{g})$

$\varphi: T\mathfrak{g} \to T\mathfrak{g}/I$.

Let $\deg(\mathfrak{g}) = 1$.

$$F_nU(\mathfrak{g}) := \varphi\Big(\bigoplus_{i=0}^{n} \mathfrak{g}^{\otimes i}\Big) \subset U(\mathfrak{g}) \quad (\subset T\mathfrak{g})$$

And we have that

$$[F_iU(\mathfrak{g}), F_jU(\mathfrak{g})] \subset F_{i+j-1}U(\mathfrak{g})$$

since $xy - yx = [x,y]$

$\Downarrow$

$gr\,U(\mathfrak{g})$ is commutative.

Thm 2.1.4 (Poincaré – Birkhoff – Witt)

There is a natural isomorphism

$$\Phi: S\mathfrak{g} \to gr\,U(\mathfrak{g})$$

$\uparrow$

Abstract version. $\Updownarrow$ $\Uparrow$ $\Phi_p: S^p\mathfrak{g} \to gr_p(\mathfrak{g})$ $\Downarrow$

Concrete version. $(I, \le)$ "$x_1 < x_2 < x_3 < \cdots$"

↓

Thm 2.1.5 (PBW) Let $\{x_1, x_2 \cdots\}$ be any ordered basis of $\mathfrak{g}$, and consider ordered monomials $\prod_i x_i^{n_i}$ where the product is ordered according to the basis.
Then such ordered monomials are linearly independent, hence form a basis of $U(\mathfrak{g})$

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Example 2.1.6 $U(\mathfrak{sl}_2)$: $e^n h^m f^k$ $n, m, k \in \mathbb{Z}_{\ge 0}$

Remark: One may consider $U(\mathfrak{g})$ as a deformation of the symmetric algebra $S\mathfrak{g}$ in the sense we consider a family of k-algebra

$$A_t := T\mathfrak{g} / \langle x \otimes y - y \otimes x - t[x, y] \mid x, y \in \mathfrak{g} \rangle$$

$A_0 \cong S\mathfrak{g}$.
$A_1 \cong U(\mathfrak{g})$

Ref: MOF - - - - -

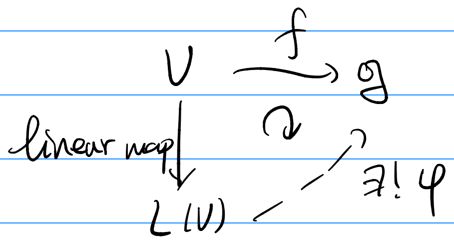
Cor 2.1.7 The map $\rho: \mathfrak{g} \to U(\mathfrak{g})$ is injective. Thus $\mathfrak{g} \subset U(\mathfrak{g})$
proof. The basis (x_i) of $\mathfrak{g}$ is mapped to linearly independent elements. $\square$

Remark V = vec space with bilinear map $[-,-] : V \times V \to V$
One can also define $U(\mathfrak{g})$ as above.
Cor 2.1.7 $\Rightarrow$ $[x, x] = 0 \ \forall x \in V$ & Jacobi identity.

The PBW Thm & Cor. 2.1.7 fail without the axiom of Lie algebra.

Let V = vec. space / $\Bbbk$ the free Lie algebra generated by V is the Lie subalgebra $L(V) \subset TV$ generated by V.

Universal $\forall \mathfrak{g}$ $f: V \to \mathfrak{g}$ linear map.



Fact. $U(L(V)) \cong TV$

§2.2. proof of PBW Thm. (V. Kac. lecture notes)

proof of Thm 2.1.5

Easy part: the ordered monomials span $U(\mathfrak{g})$

Let s be the degree of the monomial $x_{i_1} \cdots x_{i_s}$

and N the number of pairs i_m, i_n for which $m < n$ but $i_m > i_n$

(s, N)

$x_{i_1} \cdots x_7 \cdots x_3 \cdots x_{i_s}$ ($\uparrow m$, $i_m = 7$; $\uparrow n$, $i_n = 3$)

Set $(s, N) < (s', N') \iff s < s'$ or $s = s'$ & $N < N'$

Proof is by induction on the pair (s, N)

For $N = 0$, there is nothing to prove.

If $N \geq 1$, then in the monomial we have $x_{i_t} x_{i_{t+1}}$ where $i_t > i_{t+1}$

But we have the relation

$$x_{i_t} x_{i_{t+1}} = x_{i_{t+1}} x_{i_t} + [x_{i_t}, x_{i_{t+1}}]$$

Then $x_{i_1} \cdots x_{i_t} x_{i_{t+1}} \cdots x_{i_s} = \underline{x_{i_1} \cdots x_{i_{t+1}} x_{i_t} \cdots x_{i_s}} + \underline{x_{i_1} \cdots x_{i_{t-1}} [x_{i_t}, x_{i_{t+1}}] x_{i_{t+2}} \cdots x_{i_s}}$

$(s, N-1) < (s, N)$ $(s-1, ?) < (s, N)$

By induction hypothesis, each term of RHS is generated by ordered monomials.

Hard part: the ordered monomials are linearly independent.

Let B_s be the vec. space/k with a basis $y_{i_1} \cdots y_{i_s}$ where $i_1 < \cdots < i_s$ $(s>0)$

Set $B_0 = k$. $B = \bigoplus_{s \geq 0} B_s$

We shall construct a linear map $f: T_g \to B$ s.t.

$I \subset \ker f$ and $f(X_{i_1} \cdots X_{i_s}) = (y_{i_1}, \cdots, y_{i_s})$ if $i_1 < \cdots < i_s$

This will induce a linear map

$\bar{f}: U(g) \to B$

Hence the ordered monomials are linearly independent since

$y_{i_1}, \cdots, y_{i_s}$ $i_1 < \cdots < i_s$ are linearly independent.

Construction.

$f(1) = 1$. $f(X_{i_1} \cdots X_{i_s}) = (y_{i_1} \cdots y_{i_s})$ if $i_1 < \cdots < i_s$

$$f(X_{i_1} \cdots X_{i_t} X_{i_{t+1}} \cdots X_{i_s}) = f(X_{i_1} \cdots X_{i_{t+1}} X_{i_t} \cdots X_{i_s}) + f(X_{i_1} \cdots [X_{i_t}, X_{i_{t+1}}] \cdots X_{i_s}) \quad (*)$$

⇓

reduce $f(X_{i_1} \cdots X_{i_s}) = \sum f(X_{j_1} \cdots X_{j_s})$ $j_1 < \cdots < j_s$

$f(X_4 X_3 X_2 X_1) = f(X_4 X_2 X_3 X_1) + f(\cdots)$

$= f(X_4 X_2 X_1 X_3) + \cdots$ $= f(X_1 X_2 X_3 X_4) + f(\cdots [\ ] \cdots) \cdots$

$= f(X_4 X_3 X_1 X_2) + f(\cdots)$

$= f(X_4 X_1 X_3 X_2) + \cdots$

We do this by induction on (s, N)

Case 1. $X_{i_1} \cdots X_{i_s} = X_{i_1} \cdots \overset{a}{X_{i_t}} \overset{b}{X_{i_{t+1}}} \cdots \overset{c}{X_{i_r}} \overset{d}{X_{i_{r+1}}} \cdots X_{i_s}$ $\quad i_t > i_{t+1}$, $i_r > i_{r+1}$

$f(\cdots ab \cdots cd \cdots)$
$= f(\cdots ba \cdots cd \cdots) + f(\cdots [a,b] \cdots cd \cdots)$
$= f(\cdots ba \cdots dc \cdots) + f(\cdots ba \cdots [c,d] \cdots)$
$\quad (S, N-2) \qquad (S-1, N-1)$

$+ f(\cdots [a,b] \cdots dc \cdots) + f(\cdots [a,b] \cdots [c,d] \cdots)$
$\quad (S-1, N-1) \qquad (S-2, ?)$

||

$= f(\cdots ab \cdots dc \cdots) + f(\cdots ab \cdots [c,d] \cdots)$
$= f(\cdots ba \cdots dc \cdots) + f(\cdots [a,b] \cdots dc \cdots)$
$+ f(\cdots ba \cdots [c,d] \cdots) + f(\cdots [a,b] \cdots [c,d] \cdots)$

Case. $X_{i_1} \cdots X_{i_s} = X_{i_1} \cdots \overset{a}{X_{i_t}} \overset{b}{X_{i_{t+1}}} \overset{c}{X_{i_{t+2}}} \cdots X_{i_s}$ $\quad i_t > i_{t+1} > i_{t+2}$.

$f(\cdots abc \cdots) = f(\cdots bac \cdots) + f(\cdots [a,b]c \cdots)$
$= f(\cdots bca \cdots) + f(\cdots b[a,c] \cdots)$
$+ f(\cdots [a,b]c \cdots)$

$= f(\cdots cba \cdots) + f(\cdots [b,c]a \cdots)$ (1)
$+ f(\cdots b[a,c] \cdots) + f(\cdots [a,b]c \cdots)$

$= f(\cdots acb \cdots) + f(\cdots a[b,c] \cdots)$
$= f(\cdots cab \cdots) + f(\cdots [a,c]b \cdots) + f(\cdots a[b,c] \cdots)$
$= f(\cdots cba \cdots) + f(\cdots c[a,b] \cdots)$ (2)
$+ f(\cdots [a,c]b \cdots) + f(\cdots a[b,c] \cdots)$

By linearity & induction hypothesis.
$f(\cdots \alpha\beta \cdots) - f(\cdots \beta\alpha \cdots) = f(\cdots [\alpha,\beta] \cdots)$

$$(1)-(2) = f(\cdots [b,c]a \cdots) + f(\cdots b[a,c] \cdots) + f(\cdots [a,b]c \cdots)$$
$$- f(\cdots a[b,c] \cdots) - f(\cdots [a,c]b \cdots) - f(\cdots c[a,b] \cdots)$$

$$= f(\cdots [[b,c],a] \cdots) + f(\cdots [b,[a,c]] \cdots) + f(\cdots [[a,b],c] \cdots)$$

$-[a,[b,c]]$ $\qquad$ $-[b,[c,a]]$ $\qquad$ $-[c,[a,b]]$

$= 0$

$\Uparrow$ $\qquad [a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$

Jacobi identity

It remains to show that $I \subset \ker f$.

By linearity, it is sufficient to show that $\forall A, B \in Tg$, $\forall i, j$

$$f(A(X_iX_j - X_jX_i - [X_i, X_j])B) = 0$$

If $i \geq j$, this is $(*)$

$$f(X_{i_1} \cdots X_{i_t} X_{i_{t+1}} \cdots X_{i_s}) = f(X_{i_1} \cdots X_{i_{t+1}} X_{i_t} \cdots X_{i_s})$$
$$+ f(X_{i_1} \cdots [X_{i_t}, X_{i_{t+1}}] \cdots X_{i_s})$$

If $i < j$ $\quad (*) \Rightarrow f(AX_jX_iB) = f(AX_iX_jB) + f(A[X_j, X_i]B)$
$$= f(AX_iX_jB) - f(A[X_i, X_j]B)$$

$$\Rightarrow \quad f(AX_iX_jB) = f(AX_jX_iB) + f(A[X_i, X_j]B) \qquad \square$$