

Lecture 3.

§3.0 Generalized eigenspace.

§3.1 Representation of $\mathfrak{sl}_2$.

§3.2 Quantum enveloping algebra $U_q(\mathfrak{sl}_2)$

§3.3 BB localization for $\mathfrak{sl}_2$

§3.0

Def 3.0.1 $V = \text{vector space} / \mathbb{K}$ $\lambda \in \mathbb{K}$ $A: V \rightarrow V$ linear operator.

The subspace

$$V_\lambda = \{ v \in V \mid (A - \lambda I)^N v = 0 \text{ for some } N \in \mathbb{Z}_{>0} \}$$

is called a generalized eigenspace of A with eigenvalue λ

Prop. 3.0.2 $V = \text{vec. space} / \mathbb{K} = \bar{\mathbb{K}}$ $A: V \rightarrow V$ linear operator.

Let $\lambda_1, \dots, \lambda_s$ be all eigenvalues of A and $n_1, \dots, n_s$ be their multiplicities

Then there is the generalized eigenspace decomposition

$$V = \bigoplus_{i=1}^s V_{\lambda_i}$$

where $\dim V_{\lambda_i} = n_i$

proof By Jordan normal form $A = \begin{pmatrix} J_{\lambda_1} & & \\ & \ddots & \\ & & J_{\lambda_n} \end{pmatrix}$

with basis $e_1, \dots, e_{n_1}, e_{n_1+1}, \dots, e_{n_1+n_2}, \dots$

Let $V_{\lambda_1} = \text{span}(e_1, \dots, e_{n_1})$, $V_{\lambda_2} = \text{span}(e_{n_1+1}, \dots, e_{n_1+n_2})$, $\dots$ $\square$

Ref <<高等代数学>> 姚慕生 吴泉水 谢启鸿

§ 3.1. $\mathfrak{sl}_2 = \left\{ \begin{bmatrix} a & x \\ y & -a \end{bmatrix} \right\} \subset \mathfrak{gl}_2 = \text{Mat}_2(\mathbb{C})$

Recall $\mathfrak{sl}_2$ has generators $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
with relation $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$

We consider the $\mathfrak{sl}_2$ action on $V = \mathbb{C}[x, y]$ given by
 $e \mapsto x \partial_y$ $f \mapsto y \partial_x$ $h \mapsto x \partial_x - y \partial_y$

$$\begin{aligned} [x \partial_y, y \partial_x] x^l y^m &= (x \partial_y)(y \partial_x)(x^l y^m) - y \partial_x (x \partial_y x^l y^m) \\ &= (x \partial_y) l x^{l-1} y^{m+1} - (l+1) x^l y^m \\ &= l(m+1) x^l y^m - (l+1) x^l y^m \end{aligned}$$

This infinite dimension representation has the form
 $V = \bigoplus_{k \geq 0} V[k]$

where $V[k] = \{ g(x, y) \in \mathbb{C}[x, y] \mid \deg g = k \}$

$V[k]$ is invariant under e, f, h , hence it is a $k+1$ -dim representation with basis $x^k, x^{k-1}y, \dots, xy^{k-1}, y^k$ s.t.

$$\begin{aligned} e: x^l y^m &\mapsto m x^l y^{m-1} \\ f: x^l y^m &\mapsto l x^{l-1} y^{m+1} \\ h: x^l y^m &\mapsto (l-m) x^l y^m \end{aligned}$$

Remark. $V[0] \hookrightarrow$ trivial representation

$V[1] \hookrightarrow \text{GL}(V) = \text{Mat}_2$ *tautological representation by $\text{Mat}_2(\mathbb{C})$*

$V[2] \hookrightarrow$ adjoint representation

Prop. 3.1.1. $V[k]$ is irreducible.

proof Let $W \subset V[k]$ be a nonzero subrepresentation.

Therefore it is h -invariant, hence it is spanned by $\{x^n y^{k-n}\}_{n \in S}$ where $S \subset \{0, \dots, k\}$

Since W is also e -invariant and f -invariant,

if $m \in S$ then so is $m+1$ and $m-1$ (if they are in $\{0, \dots, k\}$)
Thus $S = \{0, \dots, k\}$ and $W = V[k]$ □

Prop 3.1.2 If $V \neq 0$ is a fin-dim representation of $\mathfrak{sl}_2$, then $e|_V$ and $f|_V$ are nilpotent, so $\ker(e) \neq 0$.
Moreover, h preserves $\ker(e)$ and acts diagonalizably on it, with nonnegative integer eigenvalues.

proof Write V as a direct sum of generalized eigenspaces of h .
$$V = \bigoplus_{\lambda} V_{\lambda}$$

Since $he = e(h+2)$, $hf = f(h-2)$, we have

$$e: V_{\lambda} \rightarrow V_{\lambda+2}$$

$$f: V_{\lambda} \rightarrow V_{\lambda-2}$$

Thus $e|_V, f|_V$ are nilpotent, so $\ker(e) \neq 0$. $e^N v = 0$

Take $v \in \ker(e)$, then $e(hv) = (h-2)ev = 0 \Rightarrow hv \in \ker(e)$

So $\ker(e)$ is h -invariant.

Consider $v_m := e^m f^m v$ (above $v \in \ker(e)$)

$$\begin{aligned} \text{Since } ef^m v &= (fe+h)f^{m-1}v = fe f^{m-1}v + hf^{m-1}v \\ &= fe f^{m-1}v + f^{m-1}(h-2(m-1))v \\ &= f^2 e f^{m-2}v + f^{m-1}((h-2(m-1)) + (h-2(m-2)))v \\ &= \dots \\ &= f^{m-1} m(h-m+1)v \end{aligned}$$

keep doing this.

$$\begin{aligned} \text{Then } v_m = e^m f^m v &= e^{m-1} f^{m-1} m(h-m+1)v \\ &= m(h-m+1)v_{m-1} \end{aligned}$$

Since f is nilpotent, $v_m = 0$ for large enough m .

$$\Rightarrow m!(h-m+1)(h-m)\dots(h-1)h = 0$$

$\Rightarrow hv = nv$ for some $n \in \{1, \dots, m-1\}$ } linear algebra

$\Rightarrow h$ acts diagonalizably on $\ker(e)$ with nonnegative integer eigenvalues n .

i.e. action of h on $\ker(e)$ is semisimple

$\Rightarrow$ abstract Jordan decomposition.

Thm 3.1.3 Any irred. fin-dim rep V of $\mathfrak{sl}_2$ is isomorphic to $V[k]$ for some k .

proof Let $v \in \ker(e)$ be a eigenvector of h with eigenvalue λ .

$$\text{Let } \omega_m := \frac{f^m}{m!} v, \quad \omega_m = \frac{f}{m} \omega_{m-1}$$

$$\text{Then } f \omega_m = (m+1) \omega_{m+1}$$

$$h \omega_m = (\lambda - 2m) \omega_m$$

$$e \omega_m = m(\lambda - m + 1) \omega_{m-1}$$

Prop 3.1.2 $\Rightarrow \lambda$ is a nonnegative integer. Now we write $\lambda = k$

It is easy to see that k is the maximal one such that $\omega_k \neq 0$ (i.e. $\omega_{k+1} = 0$)

$$e \omega_{k+1} = (k+1)(\lambda - k) \omega_k = 0$$

$\Rightarrow \{\omega_0, \dots, \omega_k\}$ form a basis of V and

$$\omega_m \mapsto \binom{k}{m} x^m y^{k-m} \text{ gives the isomorphism}$$

$$V \cong V[k]$$

$\square$

Thm 3.1.4 Any fin-dim rep V of $\mathfrak{sl}_2$ is completely reducible.

We will prove the complete reducibility for the rep of semisimple Lie algebra

§3.2

We first introduce the q -integers

q = variable

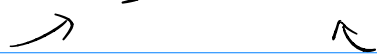
$$q\text{-integer } [n] := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \dots + q^{1-n} \in \mathbb{Q}[q, q^{-1}] =: A$$

There is a ring homomorphism

$$A \longrightarrow \mathbb{Q}$$

$$q \longmapsto 1$$

$$[n] \longmapsto n$$



q -analog

classical

Remark:

"We do not consider a specialization at roots of unity in this lecture. Suppose that $\zeta = \exp(2\pi i \frac{k}{n})$ is the primitive n th root of unity.

$$\Rightarrow \varphi: A \longrightarrow \mathbb{C}$$

$$q \longmapsto \zeta$$

$$[n] \longmapsto 0$$

$$[n+k] = [k] \text{ in the image. } \therefore \varphi([n+k]) = \varphi([k])$$

$\hookrightarrow$ similar to $\mathbb{F}_p$ if $p=n$

representation theory of QUE at $q=\zeta$ is similar to $n=p$ representation theory of reductive groups in $\text{char}=p$.

Ref: Lusztig.

Def 3.3.1 $U_q(\mathfrak{sl}_2)$ is an associative algebra / $A = \mathbb{Q}(q)$

with generators e, f, t, t^{-1}

and relations $te^{-1} = t^{-1}t = 1$

$$tft^{-1} = q^{-2}f$$

$$tet^{-1} = q^2e$$

$$[e, f] = \frac{t - t^{-1}}{q - q^{-1}}$$

$$(\leftarrow ef - fe)$$

Remark. Roughly, we may think about the element as

$$t = q^h \quad (h \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}_2) \\ \text{correspond}$$

taking limit $q \rightarrow 1$

$$\Rightarrow [e, f] = \frac{t - t^{-1}}{q - q^{-1}} = \frac{q^h - q^{-h}}{q - q^{-1}} \longrightarrow h$$

Thm 3.3.2 (PBW basis) $e^k t^n f^l \quad k, l \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}$ forms
a $\mathbb{Q}(q)$ -linear basis of $\mathcal{U}_q(\mathfrak{sl}_2)$

Hopf algebra

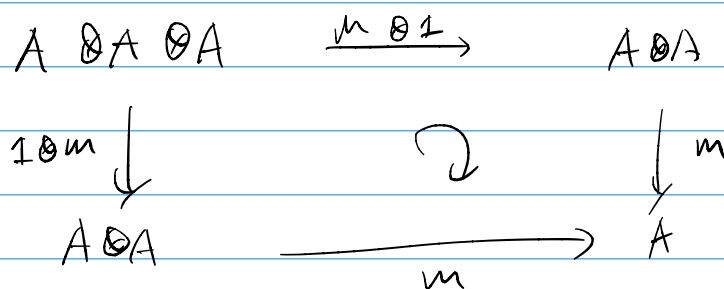
We have defined associative algebras in §2.0 which may be restated as follows.

Def 3.3.3. An (associative) algebra A over a field k is a vector space $/k$ with linear maps
 $m: A \otimes_k A \rightarrow A$ the multiplication
 and

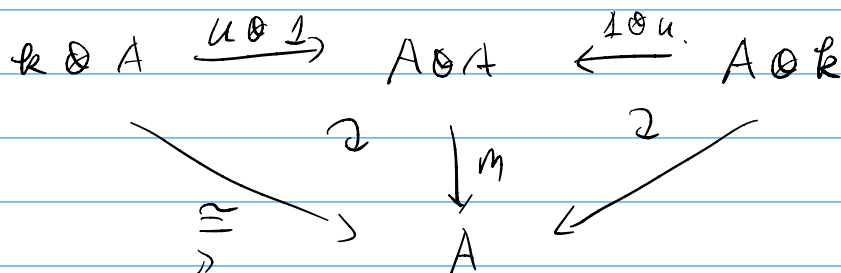
$u: k \rightarrow A$ the unit

s.t. the following diagrams are commutative.

associativity.



unit.



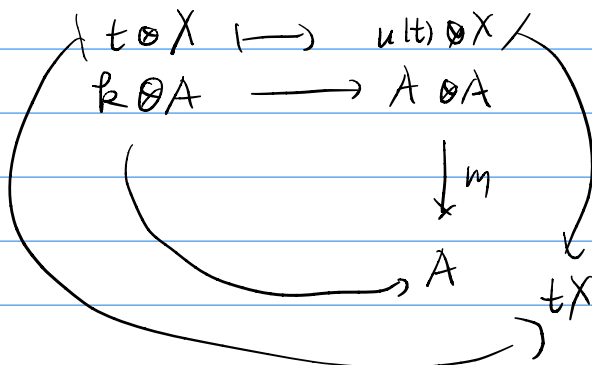
natural isomorphism

Example.

$$A = \text{Mat}_{n \times n}(k)$$

$$u: k \rightarrow A$$

$$t \mapsto tI.$$



We now defined the notion of coalgebra by reversing the direction of above arrows

Def 3.3.4 A coalgebra over k is a vector space A with linear maps

$\Delta: A \rightarrow A \otimes A$ the comultiplication.

$\varepsilon: A \rightarrow k$ the counit.

s.t. the following diagrams are commutative.

$$\begin{array}{ccccc}
 A \otimes A \otimes A & \xleftarrow{\Delta \otimes 1} & A \otimes A & & \\
 1 \otimes \Delta \uparrow & & & & \uparrow \Delta \\
 A \otimes A & \xleftarrow{\Delta} & A & & \\
 & \Delta \searrow & & & \\
 k \otimes A & \xleftarrow{\varepsilon \otimes 1} & A \otimes A & \xrightarrow{1 \otimes \varepsilon} & A \otimes k \\
 & & \uparrow \Delta & & \\
 & \cong \swarrow & A & \searrow \cong &
 \end{array}$$

Def 3.3.5 If (A, m, u) is an algebra, (A, Δ, ε) is a coalgebra and m, u are coalgebra homomorphism Δ, ε are algebra homomorphism.

And there exists a linear map

$S: A \rightarrow A$ called antipode

satisfying the following commutative diagram.

$$\begin{array}{ccccc}
 A \otimes A & \xleftarrow{\Delta} & A & \xrightarrow{\Delta} & A \otimes A \\
 S \otimes 1 \downarrow & \cong & \downarrow u \circ \varepsilon & \cong & \downarrow 1 \otimes S \\
 A \otimes A & \xrightarrow{m} & A & \xleftarrow{u} & A \otimes A
 \end{array}$$

Then $(A, m, u, \Delta, \varepsilon, S)$ is called a Hopf algebra.

Example 3.3.6 $\mathfrak{g} = \text{Lie algebra}$

(1) $A = U(\mathfrak{g})$

$$\Delta: x \mapsto \Delta(x) = x \otimes 1 + 1 \otimes x$$

$$\varepsilon: x \mapsto \varepsilon(x) = 0$$

$$S: x \mapsto S(x) = -x$$

$$\forall x \in \mathfrak{g}$$

(S is an anti algebra homomorphism)

$$\begin{array}{ccc} x \otimes 1 + 1 \otimes x & \xleftarrow{\Delta} & x \\ \downarrow S \otimes 1 & \searrow \text{c} & \downarrow \eta \circ \varepsilon \\ A \otimes A & \xrightarrow{m} & A \end{array}$$

$0 = m(-x \otimes 1 + 1 \otimes x) = -x + x = 0$

(2) $A = U_{\mathfrak{g}}(\mathfrak{sl}_2)$

$$\Delta e = e \otimes t^{-1} + 1 \otimes e$$

$$\Delta f = f \otimes 1 + t \otimes f$$

$$\Delta(t^{\pm}) = t^{\pm} \otimes t^{\pm}$$

$$\varepsilon(t) = 1 \quad \varepsilon(e) = \varepsilon(f) = 0$$

$$S(t) = t^{-1} \quad S(e) = -et \quad S(f) = -t^{-1}f$$

$U_{\mathfrak{g}}(\mathfrak{g})$ is also a Hopf algebra for general $\mathfrak{g}$ (maybe be Kac-Moody Lie algebra)

Def 3.3.7 A is commutative $\Leftrightarrow$

$$\begin{array}{ccc} a \otimes b & \xrightarrow{\quad} & b \otimes a \\ A \otimes A & \xrightarrow{\text{swap}} & A \otimes A \end{array}$$

cocommutative $\Leftrightarrow$

$$\begin{array}{ccc} A \otimes A & \xleftarrow{\text{swap}} & A \otimes A \\ \uparrow \Delta & & \uparrow \eta \circ \varepsilon \\ A & & A \end{array}$$

Fact. $U(\mathfrak{g})$ is not commutative but cocommutative

$U_{\mathfrak{g}}(\mathfrak{g})$ is neither comm nor cocomm.

§3.3 (NO PROOF)

Let us begin with some "algebraic geometry", but in this case, we can try to not use the language of sheaves.

We define the 1-dimensional project space $P^1 := (\mathbb{C}^2)^* / (a, b) \sim (\lambda a, \lambda b)$

Def 3.3.1 The second Weyl algebra

$$D_{\mathbb{C}^2} = D_2 := \mathbb{C} \langle x, y, \partial_x, \partial_y \rangle / ([x, \partial_x] = [y, \partial_y] = 1, \text{ others } = 0)$$

Def 3.3.2. A (left) $\mathcal{D}$ -module on $\mathbb{C}^2$ is a (left) D_2 -module

Actually, we have a function

$$\begin{aligned} \mathfrak{sl}_2 &\longrightarrow D_{\mathbb{C}^2} \\ e &\longmapsto x\partial_y \\ f &\longmapsto y\partial_x \\ h &\longmapsto x\partial_x - y\partial_y \end{aligned}$$

This map extends to a surjective map of associative algebras $U(\mathfrak{sl}_2) \longrightarrow D_{P^1}$

By this map, the Casimir element

$$c = \frac{1}{2}h^2 + h + 2fe$$

is sent to 0.

Fact. c generates the entire center of $U(\mathfrak{sl}_2)$

Thm 3.3.3 (Beilinson - Bernstein localization for $\mathfrak{sl}_2$)

There is an equivalence of categories between

$$\mathcal{D}\text{-Mod}(\mathbb{CP}^1) \quad \text{and} \quad \underline{U_0(\mathfrak{sl}_2)\text{-Mod}}.$$

$\mathfrak{sl}_2$ -rep with trivial central character

$$(U_0(\mathfrak{sl}_2) = U(\mathfrak{sl}_2)/\langle c = 0 \rangle)$$

The general statement:

$$G = \text{complex reductive group} \quad \mathfrak{g} = \text{Lie}(G)$$

$$B = \text{Borel subgroup} \quad G/B = \text{the flag variety}$$

$$\Gamma: \mathcal{D}\text{-Mod}(G/B) \rightarrow (\mathfrak{g}\text{-mod})_0$$

is an equivalence of categories.

Ref.

A. Beilinson, J. Bernstein Localisation de $\mathfrak{g}$ -modules. (1981)

R. Hotta, K. Takeuchi, T. Tanisaki, D-modules, perverse sheaves, and representation theory