Lecture 3.

§3.0 Generalized eigenspace.

§3.1 Representation of $\mathfrak{sl}_2$.

§3.2 Quantum enveloping algebra $U_q(\mathfrak{sl}_2)$

§3.3 BB localization for $\mathfrak{sl}_2$

§3.0

Def 3.0.1 V = vector space $/\Bbbk$ $\lambda \in \Bbbk$. $A: V \to V$ linear operator.
The subspace
$$V_\lambda = \{ v \in V \mid (A - \lambda I)^N v = 0 \text{ for some } N \in \mathbb{Z}_{>0} \}$$
is called a generalized eigenspace of A with eigenvalue λ

Prop. 3.0.2 V = vec. space $/\Bbbk = \bar{\Bbbk}$ $A: V \to V$ linear operator.
Let $\lambda_1 \cdots \lambda_s$ be all eigenvalues of A and $n_1 \cdots n_s$ be their multiplicities
Then there is the generalized eigenspace decomposition
$$V = \bigoplus_{i=1}^{s} V_{\lambda_i}$$
where $\dim V_{\lambda_i} = n_i$

proof By Jordan normal form $A = \begin{pmatrix} J_{\lambda_1} & & \\ & \ddots & \\ & & J_{\lambda_n} \end{pmatrix}$

with basis $e_1 \cdots e_{n_1}$ $e_{n_1+1} \cdots e_{n_1+n_2}$ $\cdots$
Let $V_{\lambda_1} = \mathrm{span}(e_1 \cdots e_{n_1})$, $V_{\lambda_2} = \mathrm{span}(e_{n_1+1} \cdots e_{n_1+n_2})$, $\cdots$ $\square$

Ref 《高等代数学》 姚慕生 吴泉水 谢启鸿

§ 3.1. $sl_2 = \{ \begin{bmatrix} a & * \\ * & -a \end{bmatrix} \} \subset gl_2 = Mat_2(\mathbb{C})$

Recall sl_2 has generators $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

with relation $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$

We consider the sl_2 action on $V = \mathbb{C}[x, y]$ given by

$$e \mapsto x\partial_y \quad f \mapsto y\partial_x \quad h \mapsto x\partial_x - y\partial_y$$

$[x\partial y, y\partial x] x^l y^m$
$= (x\partial y)(y\partial x)(x^l y^m) - y\partial x\, x\partial y\, x^l y^m$
$= (x\partial y)\, l x^{l-1} y^{m+1}$
$l(m+1) x^l y^m - (l+1) m\, x^l y^m$

This infinite dimension representation has the form

$$V = \bigoplus_{k \geq 0} V[k]$$

where $V[k] = \{ g(x, y) \in \mathbb{C}[x, y] \mid \deg g = k \}$

$V[k]$ is invariant under e, f, h, hence it is a $k+1$-dim representation with basis $x^k, x^{k-1}y, \cdots, xy^{k-1}, y^k$ s.t

$$e: x^l y^m \mapsto m x^{l+1} y^{m-1}$$
$$f: x^l y^m \mapsto l x^{l-1} y^{m+1}$$
$$h: x^l y^m \mapsto (l-m) x^l y^m$$

Remark. $V[0] \rightsquigarrow$ trivial representation
$V[1] \rightsquigarrow GL(V) = Mat_2$ tautological representation by $Mat_2(k)$
$V[2] \rightsquigarrow$ adjoint representation

Prop. 3.1.1 $V[k]$ is irreducible.

proof Let $W \subset V[k]$ be a nonzero subrepresentation.
Therefore it is h-invariant, hence it is spanned by $\{x^n y^{k-n}\}_{n \in S}$ where $S \subset \{0, \cdots, k\}$
Since W is also e-invariant and f-invariant,
if $m \in S$ then so is $m+1$ and $m-1$ (if they are in $\{0 \cdots k\}$)
Thus $S = \{0 \cdots k\}$ and $W = V[k]$ $\square$

Prop 3.1.2 If $V \neq 0$ is a fin-dim representation of $\mathfrak{sl}_2$, then $e|_V$ and $f|_V$ are nilpotent, so $\ker(e) \neq 0$.
Moreover, h preserves U and acts diagonalizably on it, with nonnegative integer eigenvalues.

proof Write V as a direct sum of generalized eigenspace of h.
$$V = \bigoplus V_\lambda$$
Since $he = e(h+2)$, $hf = f(h-2)$, we have
$$e: V_\lambda \to V_{\lambda+2}$$
$$f: V_\lambda \to V_{\lambda-2}$$
Thus $e|_V, f|_V$ are nilpotent, so $\ker(e) \neq 0$ $\quad e^N v = 0$

Take $v \in \ker(e)$, then $e(hv) = (h-2)ev = 0 \Rightarrow hv \in \ker(e)$
So $\ker(e)$ is h-invariant.

Consider $v_m := e^m f^m v$ (above $v \in \ker(e)$)

Since $ef^m v = (fe+h) f^{m-1} v = fef^{m-1}v + hf^{m-1}v$

keep doing this.

$$= fef^{m-1}v + f^{m-1}(h-2(m-1))v$$
$$= f^2 ef^{m-2} v + f^{m-1}((h-2(m-1)) + (h-2(m-2)))v$$
$$= \cdots$$
$$= f^{m-1} m(h-m+1)v$$
Then $v_m = e^m f^m v = e^{m-1} f^{m-1} m(h-m+1)v$
$$= m(h-m+1) v_{m-1}$$
Since f is nilpotent, $v_m = 0$ for large enough m.
$\Rightarrow m!(h-m+1)(h-m)\cdots(h-1)h = 0$
$\rightsquigarrow hv = nv$ for some $n \in \{1, \cdots, m-1\}$ } linear algebra
$\Rightarrow h$ acts diagonalizably on $\ker(e)$ with nonnegative integer eigenvalues. □
i.e. action of h on $\ker(e)$ is semisimple
$\Rightarrow$ abstract Jordan decomposition.

Thm 3.1.3 Any irrd. fin-dim rep V of sl_2 is isomorphic to $V[k]$ for some k.

proof Let $v \in \ker(e)$ be a eigenvector of h with eigenvalue λ.

Let $w_m := \frac{f^m}{m!} v$, $w_m = \frac{f}{m} w_{m-1}$

Then $f w_m = (m+1) w_{m+1}$

$h w_m = (\lambda - 2m) w_m$

$e w_m = m(\lambda - m + 1) w_{m-1}$

Prop 3.1.2 $\Rightarrow$ λ is a nonnegative integer. Now we write $\lambda = k$

It is easy to see that k is the maximal one such that $w_k \neq 0$ (i.e. $w_{k+1} = 0$)

$e w_{k+1} = (k+1)(\lambda - k) w_k = 0$

$\Rightarrow \{w_0 \cdots w_k\}$ form a basis of V and

$w_m \longmapsto \binom{n}{m} x^m y^{k-m}$ gives the isomorphism

$V \cong V[k]$ $\square$

Thm 3.1.4 Any fin-dim rep V of sl_2 is completely reducible.

We will prove the complete reducibility for the rep of semisimple Lie algebra

§3.2

We first introduce the q-intergers

q = variable

q-interger $[n] := \dfrac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \dots + q^{1-n} \in \mathbb{Z}[q, q^{-1}] =: \mathbb{A}$

There is a ring homomorphism

$$\begin{array}{ccc} \mathbb{A} & \longrightarrow & \mathbb{Z} \\ q & \longmapsto & 1 \\ [n] & \longmapsto & n \\ \nearrow & & \nwarrow \\ q\text{-analog} & & \text{classical} \end{array}$$

Remark:

" We do not consider a specialization at roots of unity in this lecture.

Suppose that $\zeta = \exp(2\pi i \frac{k}{n})$ is the primitive n^{th} root of unity.

$$\Rightarrow \varphi: \begin{array}{ccc} \mathbb{A} & \longrightarrow & \mathbb{C} \\ q & \longmapsto & \zeta \\ [n] & \longmapsto & 0 \end{array}$$

$[n+k] = [k]$ in the image : $\varphi([n+k]) = \varphi([k])$

$\rightsquigarrow$ similar to $\mathbb{F}_p$ if $p = n$

representation theory of QUE at $q = \zeta$ is similar to $n = p$ representation theory of reductive groups in char $= p$.

Ref. Lusztig.

Def 3.3.1 $U_q(sl_2)$ is an associative algebra $/ k = \mathbb{Q}(q)$

with generators e, f, t, t^{-1}

and relations $tt^{-1} = t^{-1}t = 1$ $\quad tet^{-1} = q^2 e$

$tft^{-1} = q^{-2} f$ $\quad [e, f] = \dfrac{t - t^{-1}}{q - q^{-1}}$

$(= ef - fe)$

Remark. Roughly, we may think about the element as
$t = q^h$ $\qquad$ ($h \underset{\text{correspond}}{\leftrightarrow} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in sl_2$)

taking limit $q \to 1$

$$\Rightarrow [e, f] = \frac{t - t^{-1}}{q - q^{-1}} = \frac{q^h - q^{-h}}{q - q^{-1}} \longrightarrow h.$$

Thm 3.3.2 (PBW basis) $e^k t^n f^l$ $\quad k, l \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}$, forms a $\mathbb{Q}(q)$-linear basis of $U_q(sl_2)$

Hopf algebra

We have defined associative algebras in § 2.0 which may be restated as follows.

Def 3.3.3. An (associative) algebra A over a field k is a vector space $/k$ with linear maps

$$m: A \otimes_k A \longrightarrow A \qquad \text{the multiplication}$$

and

$$u: k \longrightarrow A \qquad \text{the unit}$$

s.t. the following diagrams are commutative.

associativity.

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "m \otimes 1"] \arrow[d, "1 \otimes m"'] \arrow[dr, phantom, "\circlearrowright"] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"'] & A
\end{tikzcd}$$

unit.

$$\begin{tikzcd}
k \otimes A \arrow[r, "u \otimes 1"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "m"] & A \otimes k \arrow[l, "1 \otimes u"'] \arrow[dl] \\
& A &
\end{tikzcd}$$

natural isomorphism

Example. $A = Mat_{n \times n}(k)$ $\qquad u: k \longrightarrow A$, $\quad t \longmapsto t I$.

$$\begin{tikzcd}
\{ t \otimes X \arrow[r, mapsto] \arrow[rrd, bend right, mapsto] & u(t) \otimes X \} \\
k \otimes A \arrow[r] \arrow[rd, bend right] & A \otimes A \arrow[d, "m"] \\
& A & tX
\end{tikzcd}$$

We now defined the notion of coalgebra by reversing the direction of above arrows

Def 3.3.4 A coalgebra over k is a vector space$/k$ with linear maps

$\Delta: A \longrightarrow A \otimes A$ the comultiplication.

$\varepsilon: A \longrightarrow k$ the counit.

s.t. the following diagrams are commutative.

$$\begin{tikzcd}
A \otimes A \otimes A & & A \otimes A \arrow[ll, "\Delta \otimes 1"'] \\
A \otimes A \arrow[u, "1 \otimes \Delta"] & & A \arrow[ll, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

$$\begin{tikzcd}
k \otimes A & A \otimes A \arrow[l, "\varepsilon \otimes 1"'] \arrow[r, "1 \otimes \varepsilon"] & A \otimes k \\
& A \arrow[ul, "\cong"] \arrow[u, "\Delta"'] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

Def 3.3.5 If (A, m, u) is an algebra, (A, Δ, ε) is a coalgebra and

m, u are coalgebra homomorphism

Δ, ε are algebra homomorphism.

And there exists a linear map

$S: A \to A$ called antipode

satisfying the following commutative diagram.

$$\begin{tikzcd}
A \otimes A \arrow[d, "S \otimes 1"'] & A \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "u \circ \varepsilon"] & A \otimes A \arrow[d, "1 \otimes S"] \\
A \otimes A \arrow[r, "m"'] & A & A \otimes A \arrow[l, "m"]
\end{tikzcd}$$

Then $(A, m, u, \Delta, \varepsilon, S)$ is called a Hopf algebra.

Example 3.3.6 $\mathfrak{g}$ = Lie algebra

(1) $A = U(\mathfrak{g})$

$$\begin{aligned}
\Delta &: x \longmapsto \Delta(x) = x \otimes 1 + 1 \otimes x \\
\varepsilon &: x \longmapsto \varepsilon(x) = 0 \qquad\qquad \forall x \in \mathfrak{g} \\
S &: x \longmapsto S(x) = -x
\end{aligned}$$

(S is an anti algebra homomorphism)

$$\begin{tikzcd}
x\otimes 1 + 1 \otimes x & & x \arrow[ll, mapsto] \\
A\otimes A \arrow[d, "S\otimes 1"'] & & A \arrow[ll, "\Delta"'] \arrow[d, "u\circ\varepsilon"] \arrow[ll, phantom, bend left=0, ""] \\
A\otimes A \arrow[rr, "m"'] & & A
\end{tikzcd}$$

$S(x)\otimes 1 + 1 \otimes x \longmapsto 0 = m(-x\otimes 1 + 1 \otimes x)$

$= -x + x = 0$

(2) $A = U_q(sl_2)$

$\Delta e = e \otimes t^{-1} + 1 \otimes e$

$\Delta f = f \otimes 1 + t \otimes f$

$\Delta(t^{\pm}) = t^{\pm} \otimes t^{\pm}$

$\varepsilon(t) = 1 \quad \varepsilon(e) = \varepsilon(f) = 0$

$S(t) = t^{-1} \quad S(e) = -et \quad S(f) = -t^{-1} f$.

$U_q(\mathfrak{g})$ is also a Hopf algebra for general $\mathfrak{g}$ (might be Kac-Moody Lie algebra)

Def 3.3.7 A is commutative $\iff$

$$\begin{tikzcd}
a\otimes b \arrow[r, mapsto] & b\otimes a \\
A\otimes A \arrow[r, "\mathrm{swap}"] \arrow[dr, "m"'] & A\otimes A \arrow[d, "m"] \\
& A
\end{tikzcd}$$

cocommutative $\iff$

$$\begin{tikzcd}
A\otimes A & A\otimes A \arrow[l, "\mathrm{swap}"'] \\
& A \arrow[ul, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

Fact. $U(\mathfrak{g})$ is not commutative but cocommutative

$U_q(\mathfrak{g})$ is neither comm, nor cocomm.

§3.3 (NO PROOF)

Let us begin with some "algebraic geometry", but in this case, we can try to not use the language of sheaves.

We define the 1-dimensional project space $\mathbb{P}^1 := (\mathbb{C}^2)^\times / (a,b) \sim (\lambda a, \lambda b)$

Def 3.3.1 The second Weyl algebra

$$D_{\mathbb{C}^2} = D_2 := \mathbb{C}\langle x, y, \partial_x, \partial_y \rangle \Big/ \begin{array}{l} ([x,\partial_x] = [y,\partial_y] = 1 \\ \text{others} = 0 \end{array}$$

Def 3.3.2. A (left) $\mathcal{D}$-module on $\mathbb{C}^2$ is a (left) D_2-module

Actually, we have a function

$$\begin{aligned} sl_2 &\longrightarrow D_{\mathbb{C}^2} \\ e &\longmapsto x\partial_y \\ f &\longmapsto y\partial_x \\ h &\longmapsto x\partial_x - y\partial_y \end{aligned}$$

This map extend to a surjective map of associative algebras

$$U(sl_2) \longrightarrow D_{\mathbb{P}^1}$$

By this map, the Casimir element

$$c = \frac{1}{2}h^2 + h + 2fe$$

is sent to 0.

Fact. c generates the entire center of $U(sl_2)$

Thm 3.3.3 (Beilinson – Bernstein localization for $\mathfrak{sl}_2$)
There is an equivalence of categories between
$\mathcal{D}\text{-Mod}(\mathbb{P}^1)$ and $U_0(\mathfrak{sl}_2)\text{-Mod}$.

$\mathfrak{sl}_2$-rep with trivial central character

$(U_0(\mathfrak{sl}_2) = U(\mathfrak{sl}_2)/(c=0))$

The general statement:
G = complex reductive group $\qquad \mathfrak{g} = \mathrm{Lie}(G)$
$\cup$
B = Borel subgroup $\qquad G/B$ = the flag variety

$\Gamma: \mathcal{D}\text{-Mod}(G/B) \longrightarrow (\mathfrak{g}\text{-mod})_0$
is an equivalence of categories.

Ref.

A. Beilinson, J. Bernstein Localisation de $\mathfrak{g}$-modules. (1981)
R. Hotta, K. Takeuchi, T. Tanisaki, D-modules, perverse sheaves, and representation theory