Symmetric Polynomials

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October 11, 2022

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1 The Ring of symmetric functions

Consider the ring $\mathbb{Z}[x_1, \dots, x_n]$ of polynomials in n independent variables $x_1, \dots, x_n$ with rational integer coefficients. The symmetric group S_n acts on this ring by permuting the variables, and a polynomial is symmetric if it is invariant under this action. The symmetric polynomials form a subring

$$\Lambda_n = \mathbb{Z}[x_1, \dots, x_n]^{S_n}.$$

Λ_n is a graded ring: we have

$$\Lambda_n = \bigoplus_{k \geqslant 0} \Lambda_n^k$$

where Λ_n^k consists of the homogeneous symmetric polynomials of degree k, together with the zero polynomial.

For each $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we denote by x^α the monomial

$$x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}.$$

Let λ be any partition of length $\leqslant n$. The polynomial

$$m_\lambda(x_1, \dots, x_n) = \sum x^\alpha \tag{1}$$

summed over all distinct permutations α of $\lambda = (\lambda_1, \dots, \lambda_n)$, is clearly symmetric, and the m_λ (as λ runs through all partitions of length $\leqslant n$) form a $\mathbb{Z}-$basis of Λ_n. Hence the m_λ such that $l(\lambda) \leqslant n$ and $|\lambda| = k$ form a $\mathbb{Z}-$basis of Λ_n^k; in particular, as soon as $n \geqslant k$, the m_λ such that $|\lambda| = k$ form a $\mathbb{Z}-$basis of Λ_n^k.

I will supply a proof that the (m_λ) consists a $\mathbb{Z}-$basis of Λ_n.

Proof. We proof by induction on the degree of the symmetric polynomial f.

When $deg f = 1$, $f = a(x_1 + \dots + x_n) = a m_\lambda$ for some nonzero $a \in \mathbb{Z}$ and $\lambda = (1, 0, \dots, 0)$.

Now choose the monomial x^λ in f such that λ is maximal in lexicographic ordering and the coefficient $a_\lambda \neq 0$. Assume $|\lambda| = k+1$. Since f is symmetric, $a_\lambda m_\lambda$ is in f. Substract $a_\lambda m_\lambda$ form f we then get a symmetric polynomial with smaller lexicographic ordering. Continue this process and after finite steps, we can subtract all the $a_\lambda m_\lambda$ in f such that $|\lambda| = k + 1$. We get a symmetric polynomial g of degree k, and the induction shows that g is a finite linear combination of m_λ with integer coefficients and $|\lambda| \leqslant k$. Thus, f is a finite linear combimation of m_λ. We thus see (m_λ) indexed by all the partitions spans Λ_n over $\mathbb{Z}$. That they are linearly independent follows from the fact that these n variables $x_1, \ldots, x_n$ are algebraically independent. $\square$

In the theory of symmetric functions, the number of variables is usually irrelevant, provided only that it is large enough, and it is often more convenient to work with symmetric functions in infinitely many variables. To make this idea precise, let $m \geqslant n$ and consider the homomorphism

$$\mathbb{Z}[x_1, \ldots, x_n] \to \mathbb{Z}[x_1, \ldots, x_n]$$

which sends each of $x_{n+1}, \ldots, x_m$ to zero and other x_i to themselves. On restriction to Λ_m this gives a homomorphism

$$\rho_{m,n} : \Lambda_m \to \Lambda_n$$

whose effect on the basis (m_λ) is easily dascribed; it sends $m_\lambda(x_1, \ldots, x_m)$ to $m_\lambda(x_1, \ldots, x_n)$ if $l(\lambda) \leqslant n$, and to 0 if $l(\lambda) > n$. It follows that $\rho_{m,n}$ is surjective. On restriction to Λ_m^k we have homomorphisms

$$\rho_{m,n} : \Lambda_m^k \to \Lambda_n^k$$

for all $k \geqslant 0$ and $m \geqslant n$, which are always surjective, and are bijective for $m \geqslant n \geqslant k$.

We now form the inverse limit

$$\Lambda^k = \lim_{\substack{\longleftarrow \\ i}} \Lambda^k_n$$

of the $\mathbb{Z}-$modules Λ^k_n relative to the homomorphisms $\rho^k_{m,n}$: an element of Λ^k is by definition a sequence $f = (f_n)_{n \geqslant 0}$, where each $f_n = f_n(x_1, \ldots, x_n)$ is a homogeneous symmetric polynomial of degree k in $x_1, \ldots, x_n$, and $f_m(x_1, \ldots, x_n, 0, \ldots, 0) = f_n(x_1, \ldots, x_n)$ whenever $m \geqslant n$. Since $\rho^k_{m,n}$ is an isomorphism for $m \geqslant n \geqslant k$, it follows that the projection

$$\rho^k_n : \Lambda^k \to \Lambda^k_n,$$

which sends f to f_n, is an isomorphism for all $n \geqslant k$, and hence that Λ^k has a $\mathbb{Z}-$basis consisting of the monomial symmetric functions m_λ (for all partitions λ of k) defined by

$$\rho^k_n(m_\lambda) = m_\lambda(x_1, \ldots, x_n)$$

for all $n \geqslant k$. Hence Λ^k is a free $\mathbb{Z}-$module of rank $p(k)$, the number of partitions of k.

Now let

$$\Lambda = \bigoplus_{k \geqslant 0} \Lambda^k,$$

so that Λ is the free $\mathbb{Z}-$module generated by the m_λ for all partitions λ. We have surjective homomorphism

$$\rho_n = \bigoplus_{k \geqslant 0} \rho^k_n : \Lambda \to \Lambda_n$$

for each $n \geqslant 0$, and ρ_n is an isomorphism in degrees $k \leqslant n$.

It is clear that Λ has a structure of a graded ring such that the ρ_n are ring homomorphisms. The graded ring Λ thus defined is called the ring of symmetric functions in countably many independent variables $x_1, x_2, \ldots,$

2 Elementary symmetric functions

For each integer $r \geqslant 0$ the rth elementary symmetric function e_r is the sum of all products of r distinct variables x_i, so that $e_0 = 1$ and

$$e_r = \sum_{i_1 < i_2 < \ldots < i_r} x_{i_1} x_{i_2} \ldots x_{i_r} = m_{(1^r)}$$

for $r \geqslant 1$. The generating function for the e_r is

$$E(t) = \sum_{r \geqslant 0} e_r t^r = \prod_{i \geqslant 1} (1 + x_i t) \tag{2}$$

(t being another variable), as one sees by multiplying out the product on the right. (If the number of variables is finite, say n, then e_r)(i.e. $\rho_n(e_r)$) is zero for all $r > n$, and (2) then takes the form

$$\sum_{r=0}^{n} e_r t^r = \prod_{i=1}^{n} (1 + x_i t),$$

both sides now being elements of $\Lambda_n[t]$.

For each partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ define

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \ldots$$

Let λ be a partition, $\lambda^{'}$ be its conjugate. Then

$$e_{\lambda'} = m_\lambda + \sum_{\mu} a_{\lambda\mu} m_\mu \tag{3}$$

where the $a_{\lambda\mu}$ are non-negative integers, and the sum is over partitions $\mu < \lambda$ in the natural ordering.

Proof. When we multiply out the product $e_{\lambda'} = e_{\lambda_1'} e_{\lambda_2'} \ldots$, we shall obtain a sum of monomials, each of which is of the form

$$(x_{i_i} x_{i_2} \ldots)(x_{j_1} x_{j_2} \ldots) \ldots = x^\alpha,$$

say, where $i_1 < i_2 < \ldots < i'_{\lambda_1}, j_1 < j_2 < \ldots < j'_{\lambda_2}$, and so on. If we now enter the numbers $i_1, i_2, \ldots, i'_{\lambda_1}$ in order down the first column of the diagram of λ, then the numbers $j_1, j_2, \ldots, j'_{\lambda_2}$ in order down the second column, and so on, it is clear that for each $r \geqslant 1$ all the symbols $\leqslant r$ so entered in the diagram of λ must occur in the top r rows. Hence $\alpha_1 + \ldots + \alpha_r \leqslant \lambda_1 + \ldots + \lambda_r$ for each $r \geqslant 1$, i.e. we have $\alpha \leqslant \lambda$. Hence,

$$e_{\lambda'} = \sum_{\mu \leqslant \lambda} a_{\lambda\mu} m_\mu$$

with $a_{\lambda\mu} \geqslant 0$ for each $\mu \leqslant \lambda$, and the argument above also shows that the monomial x^λ occurs only once, so that $a_{\lambda\lambda} = 1$.

□

We have

$$\Lambda = \mathbb{Z}[e_1, e_2, \ldots] \tag{4}$$

and the e_r are algebraically independent over $\mathbb{Z}$.

Proof. The m_λ form a $\mathbb{Z}-$basis of Λ, and (3) shows that the e_λ form another $\mathbb{Z}-$basis: in other words, every element of Λ of Λ is uniquely expressible as a polynomial in the e_r. □

When there are only finitely many variables $x_1, \ldots, x_n$, (4) states that $\Lambda_n = \mathbb{Z}[e_1, \ldots, e_n]$, and that $e_1, \ldots, e_n$ are algebraically independent. This is the usual statement of the "fundamental theorem on symmetric polynomials".

3 Complete symmetric functions

For each $r \geqslant 0$ the rth complete symmetric function h_r is the sum of all monomials of total degree r in the variables $x_1, x_2, \ldots$, so that

$$h_r = \sum_{|\lambda|=r} m_\lambda$$

In particular, $h_0 = 1$ and $h_1 = e_1$. It is convenient to define h_r and e_r to be zero for $r < 0$.

The generating function for the h_r is

$$H(t) = \sum_{r \geqslant 0} h_r t^r = \prod_{i \geqslant 1} (1 - x_i t)^{-1}. \tag{5}$$

To see this, observe that

$$(1 - x_i t)^{-1} = \sum_{k \geqslant 0} x_i^k t^k,$$

and multiply these geometric series together.

From (2) and (5) we have

$$H(t)E(-t) = 1 \tag{6}$$

or, equivalently,

$$\sum_{r=0}^{n} (-1)^r e_r h_{n-r} = 0 \tag{7}$$

for all $n \geqslant 1$.

Since the e_r are algebraically independent, we may define a homomorphism of graded rings

$$\omega : \Lambda \to \Lambda$$

$$e_r \mapsto h_r$$

Using (7) and mathematical induction we can see $\omega(h_r) = e_r$ for $r \geqslant 0$. This shows that ω is an involution.

It follows that ω is an automorphism of Λ, and hence from (4) that

$$\Lambda = \mathbb{Z}[h_1, h_2, \ldots] \tag{8}$$

and the h_r are algebraically independent over $\mathbb{Z}$.

As in the case of the $e^{'}$s, we define

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \ldots$$

for any partition $\lambda = (\lambda_1, \lambda_2, \ldots)$. By (8), the h_λ form a $\mathbb{Z}-$basis of Λ. We now have three $\mathbb{Z}-$basis, all indexed by partitions: the m_λ, the e_λ, and the h_λ, the last two of which correspond under the involution ω. If we define

$$f_\lambda = \omega(m_\lambda)$$

for each partition λ, the f_λ form a fourth $\mathbb{Z} - basis$ of Λ.

4 Power sums

For each $r \geqslant 1$ the rth power sum is

$$p_r = \sum x_i^r = m_{(r)}.$$

The generating function for the p_r is

$$\begin{aligned} P(t) &= \sum_{r \geqslant 1} p_r t^{r-1} = \sum_{i \geqslant 1} \sum_{r \geqslant 1} x_i^r t^{r-1} \\ &= \sum_{i \geqslant 1} \frac{x_i}{1 - x_i t} \\ &= \sum_{i \geqslant 1} \frac{d}{dt} log \frac{1}{1 - x_i t} \end{aligned}$$

so that

$$P(t) = \frac{d}{dt} log \prod_{i \geqslant 1} (1 - x_i t)^{-1} = \frac{d}{dt} log H(t) = \frac{H^{'}(t)}{H(t)}. \tag{9}$$

Likewise we have

$$P(-t) = \frac{d}{dt} log E(t) = \frac{E^{'}(t)}{E(t)} \tag{10}$$

From (9) and (10) we obtain

$$nh_n = \sum_{r=1}^{n} p_r h_{n-r} \tag{11}$$

$$ne_n = \sum_{r=1}^{n} (-1)^{r-1} p_r e_{n-r} \tag{12}$$

for $n \geqslant 1$, and these equations enable us to express the $h^{'}$s and the $e^{'}$s in terms of the $p^{'}$s, and vice versa. The equation (12) are due to Issac Newton, and are known as Newton's formulas. From (11) it is clear that $h_n \in \mathbb{Q}[p_1, \ldots, p_n]$ and $p_n \in \mathbb{Z}[h_1, \ldots, h_n]$, and hence that

$$\mathbb{Q}[p_1, \ldots, p_n] = \mathbb{Q}[h_1, \ldots, h_n].$$

Since the h_r are algebraically independent over $\mathbb{Z}$, and hence also over $\mathbb{Q}$, it follows that

$$\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[p_1, p_2, \ldots] \tag{13}$$

and the p_r are algebraically independent over $\mathbb{Q}$.

Hence, if we define

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$$

for each partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, then the p_λ form a $\mathbb{Q}-$basis of $\Lambda_{\mathbb{Q}}$.

Since the involution ω interchanges $E(t)$ and $H(t)$ it follows from (11) and (12) that

$$\omega(p_n) = (-1)^{n-1} p_n$$

for all $n \geqslant 1$, and hence that for any partition λ we have

$$\omega(p_\lambda) = \epsilon_\lambda p_\lambda \tag{14}$$

where $\epsilon_\lambda = (-1)^{|\lambda| - l(\lambda)}$.

Finally, we shall express h_n and e_n as linear combinations of the p_λ. For any partition λ, define

$$z_\lambda = \prod_{i \geqslant 1} i^{m_i} m_i!$$

where $m_i = m_i(\lambda)$ is the number of entries of λ equal to i. Then we have

$$H(t) = \sum_\lambda z_\lambda^{-1} p_\lambda t^{|\lambda|} \tag{15}$$

$$E(t) = \sum_\lambda \epsilon_\lambda z_\lambda^{-1} p_\lambda t^{|\lambda|} \tag{16}$$

or equivalently,

$$h_n = \sum_{|\lambda|=n} z_\lambda^{-1} p_\lambda \tag{17}$$

$$e_n = \sum_{|\lambda|=n} \epsilon_\lambda z_\lambda^{-1} p_\lambda \tag{18}$$

Proof. It is enough to prove the the identity (15), since the identity (16) then follows by applying the involution ω and using (14). From (9) we have

$$\begin{aligned} H(t) &= exp \sum_{r \geqslant 1} p_r t^r / r \\ &= \prod_{r \geqslant 1} exp(p_r t^r / r) \\ &= \prod_{r \geqslant 1} \sum_{m_r = 0}^{\infty} (p_r t^r)^{m_r} / r^{m_r} m_r! \\ &= \sum_\lambda z_\lambda^{-1} p_\lambda t^{|\lambda|} \end{aligned}$$

□

Denote $(p_r t^r)^{m_r}/r^{m_r}m_r!$ by $A(r, m_r)$, then

$$\prod_{r\geqslant 1}\sum_{m_r=0}^{\infty}(p_r t^r)^{m_r}/r^{m_r}m_r!$$

$$\begin{aligned}=&(A(0,0)+A(0,1)+A(0,2)+A(0,3)+\cdots)\cdot\\ &(A(1,0)+A(1,1)+A(1,2)+A(1,3)+\cdots)\cdot\\ &(A(2,0)+A(2,1)+A(2,2)+A(2,3)+\cdots)\cdot\\ &(A(3,0)+A(3,1)+A(3,2)+A(3,3)+\cdots)\cdots\end{aligned}$$

5 Bilinear form and orthogonality

Let $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ be two finite or infinite sequences of independent variables. We shall denote the symmetric functions of the $x^{'}$s by $s_\lambda(x), p_\lambda(x)$, etc., and the symmetric function of $y^{'}$s by $s_\lambda(y), p_\lambda(y)$, etc.

We shall give there series expansions for the product

$$\prod_{i,j}(1-x_iy_j)^{-1}.$$

The first of these is

$$\prod_{i,j}(1-x_iy_j)^{-1}=\sum_{\lambda}z_\lambda^{-1}p_\lambda(x)p_\lambda(y) \tag{19}$$

summed over all partitions λ.

Proof. Let $h_r = \sum_{|\lambda|=r} m_\lambda$ be the complete symmetric functions in the variables x_iy_j. The generating function is $H(t) = \sum_{r\geqslant 0} h_r t^r = \prod_{i,j\geqslant 1}(1-x_iy_jt)^{-1}$. By (15), $H(t) = \sum_{\lambda} z_\lambda^{-1}p_\lambda t^{|\lambda|}$. Letting $t = 1$ gives

$$\prod_{i,j\geqslant 1}(1-x_iy_j)^{-1}=\sum_{\lambda}z_\lambda^{-1}p_\lambda(x_iy_j)$$

for $k \in \mathbb{Z}_{\geqslant 0}$, $p_k(x_i y_j) = p_k(x_i)p_k(y_j)$, then the result follows. $\square$

Next we have

$$\prod_{i,j}(1-x_iy_j)^{-1} = \sum_\lambda h_\lambda(x)m_\lambda(y) = \sum_\lambda m_\lambda(x)h_\lambda(y) \tag{20}$$

summed over all partitions λ.

Proof. We have

$$\begin{aligned}\prod_{i,j}(1-x_iy_j)^{-1} &= \prod_j H(y_j)\\ &= \prod_j \sum_{r=0}^{\infty} h_r(x)y_j^r\\ &= \sum_\alpha h_\alpha(x)y^\alpha\\ &= \sum_\lambda h_\lambda(x)m_\lambda(y)\end{aligned}$$

where α runs through all sequences $(\alpha_1, \alpha_2, \ldots)$ of non-negative integers such that $\sum \alpha_i < \infty$, and λ runs through all partitions. $\square$

The third identity is

$$\prod_{i,j}(1-x_iy_j)^{-1} = \sum_\lambda s_\lambda(x)s_\lambda(y) \tag{21}$$

summed over all partitions λ.

Proof. We first prove (21) for n variables x_i and n variables y_j; then letting $n \to \infty$ as usual. $\square$

We now define a scalar product on Λ, i.e. a $\mathbb{Z}-$valued bilinear form $\langle u, v\rangle$, by requiring that the bases (h_λ) and (m_λ) should be dual to each other:

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu} \tag{22}$$

for all partitions λ, μ, where $\delta_{\lambda,\mu}$ is the Kronecker delta.

For each $n \geqslant 0$, let $(u_\lambda), (v_\lambda)$ be $\mathbb{Q}-$bases of $\Lambda^n_{\mathbb{Q}}$, indexed by the partitions of n. Then the following conditions are equivalent:

$$\langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu} \text{ for all } \lambda, \mu \tag{23}$$

$$\sum_\lambda u_\lambda(x) v_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1} \tag{24}$$

Proof. Let

$$u_\lambda = \sum_\rho a_{\lambda\rho} h_\rho \quad v_\mu = \sum_\sigma b_{\mu\sigma} m_\sigma.$$

Then

$$\langle u_\lambda, v_\mu \rangle = \sum_\rho a_{\lambda\rho} b_{\mu\rho}$$

so that (23) is equivalent to

$$\sum_\rho a_{\lambda\rho} b_{\mu\rho} = \delta_{\lambda\mu}. \tag{25}$$

Also (24) is equivalent to the identity

$$\sum_\lambda u_\lambda(x) v_\lambda(y) = \sum_\rho h_\rho(x) m_\rho(y) \tag{26}$$

by (20), hence is equivalent to

$$\sum_\lambda a_{\lambda\rho} b_{\lambda\sigma} = \delta_{\rho\sigma}. \tag{27}$$

We use A to denote the matrix $(a_{\lambda\mu})$ and B to denote the matrix $(b_{\lambda\mu})$. Then (25) is equivalent to

$$A \cdot B^T = I \tag{28}$$

Also (27) is equivalent to

$$B^T \cdot A = I \tag{29}$$

Since (28) and (29) are equivalent, so are (23) and (24). $\square$

From (23) (24) and (19) it follows that

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda \tag{30}$$

so that the p_λ form an orthogonal basis of $\Lambda_{\mathbb{Q}}$. Likewise (23) (24) and (21) we have

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu} \tag{31}$$

so that the s_λ form an orthonormal basis of Λ, and the s_λ such that $|\lambda| = n$ form an orthonormal basis of Λ^n. Any other orthonormal basis of Λ^n must therefore be obtained from the basis (s_λ) by transformation by an orthogonal integer matrix. The only such matrices are signed permutation matrices, and therefore (31) characterizes the s_λ, up to order and sign.

Also from (30) and (31) we see that

The bilinear form $\langle u, v \rangle$ is symmetric and positive definite. (32)

The involution ω is an isometry, i.e. $\langle \omega u, \omega v \rangle = \langle u, v \rangle$. (33)

Proof. From (14) we have $\omega(p_\lambda) = \pm p_\lambda$, hence by (30)

$$\langle \omega(p_\lambda), \omega(p_\mu) \rangle = \langle p_\lambda, p_\mu \rangle$$

which proves (33), since the p_λ form a $\mathbb{Q}-$basis of $\Lambda_{\mathbb{Q}}$. $\square$