Claim 1: The product operation makes the set of tableaux into an associative monoid.

The empty tableau is the unit element

Claim 2: Starting with a given skew tableau, all choices of inside corners lead to the same rectified tableau.

Claim 3: The 2 product coincide.

$$T \cdot_1 U = T \cdot_2 U$$

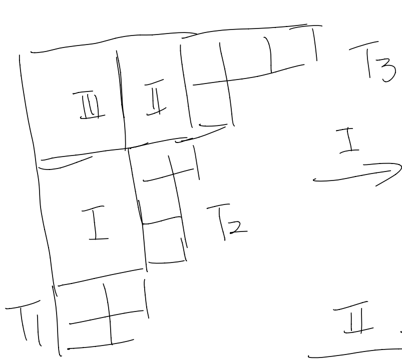
$$\text{claim 2} + \text{claim 3} \Rightarrow \text{Claim 1}$$

$$(T_1 \cdot T_2) \cdot T_3 = T_1 \cdot (T_2 \cdot T_3)$$

skew tableau

$$(T_1 \cdot_1 T_2) \cdot_1 T_3$$

$\Downarrow$ Young tableau



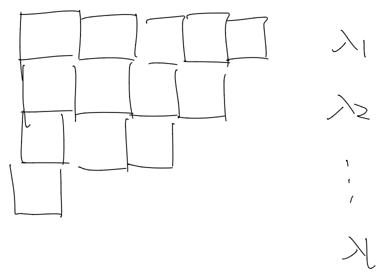
$$\xrightarrow{\mathrm{I}} (T_1 \cdot_2 T_2) * T_3 \xrightarrow{\mathrm{II}+\mathrm{III}} (T_1 \cdot_2 T_2) \cdot_2 T_3$$

$\Updownarrow$ Young tableaux

$$\xrightarrow{\mathrm{II}} T_1 * (T_2 \cdot_2 T_3) \xrightarrow{\mathrm{I}+\mathrm{III}} T_1 \cdot_2 (T_2 \cdot_2 T_3)$$

$T_1 \cdot (T_2 \cdot T_3)$

Young diagram : a collections of boxes arranged in left-justified rows with a weakly decreasing number of boxes in each row

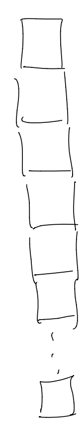


Young tableaux

n = total number of boxes

$n = \lambda_1 + \lambda_2 + \cdots + \lambda_l$ partition of n

$$n = \underbrace{1 + 1 + 1 + \cdots + 1}_{n \text{ copies}}$$



numbering or filling of the diagram

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$$

<u>filling</u>

A Young tableau, is a filling s.t

i) the numbers are weakly increasing across each row

ii) the numbers are strictly increasing across each column

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 3 & 5 \\ \hline 4 & 4 & 6 & \\ \cline{1-3}\end{array} \qquad \underline{(4,4,3)}$$

transpose

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 1 & 3 & 4 \\ \hline 2 & 3 & 6 \\ \hline 3 & 5 & \\ \cline{1-2}\end{array} \qquad (3,3,3,2)$$

$\underline{\lambda}$, m $\qquad (\lambda_1, \lambda_2, \dots, \lambda_m)$

$$\begin{array}{|c|c|c|l}\hline 1 & 2 & 3 & \lambda_1 \\ \hline 3 & 4 & 5 & \lambda_2 \\ \hline 5 & m & & \\ \cline{1-2} & & & \lambda_m \end{array}$$

Schur polynomial ; $\underline{x^T}$

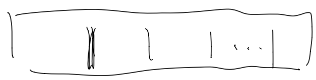
$$x_1^1 . x_2^1 . x_3^2 \quad x_m^1$$

$$x^T = \prod_{i=1}^{m} (x_i)^{\text{number of times that } i \text{ } \cancel{\text{to}} \text{ occurs in } T}$$

$$S_\lambda(x_1,\ldots,x_m) = \sum_{T \text{ is tableau}} x^T$$

T of shape λ using numbers from 1 to m

$\lambda = (n)$

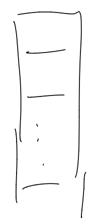


$1 \ldots m$

$$S_\lambda = \sum_{x \text{ is of degree } n} x^T = h_n(x_1,\ldots,x_m)$$

complete symmetric polynomial

$\lambda = (1, 1, \ldots, 1)$



elementary symmetric poly

$$S_\lambda = \sum x^T = e_n(x_1,\ldots,x_m)$$

$x^T = x_{i_1} \cdot \ldots \cdot x_{i_n} \qquad 1 \leq i_1 < i_2 < \ldots < i_n \leq m$

$R[x_1,\ldots,x_m]$

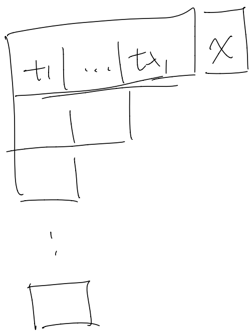
$$\underline{f(x_1,\ldots,x_m)} = P(e_1,\ldots,e_m)$$

row-insertion / row bumping

sliding / digging a hole

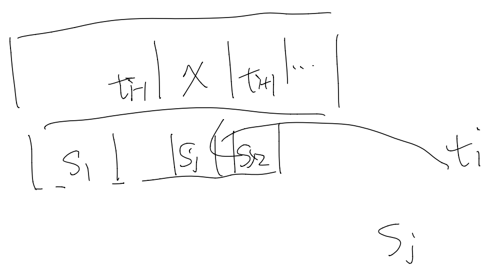
$x \in \mathbb{Z}_{>0}$

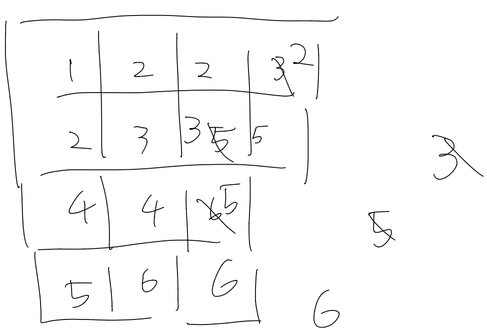
$T \leftarrow x$



$\underline{t_1, \dots, t_{\lambda_1} \le x}$

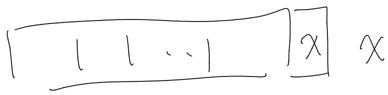
$\underline{T} \leftarrow \underline{x}$

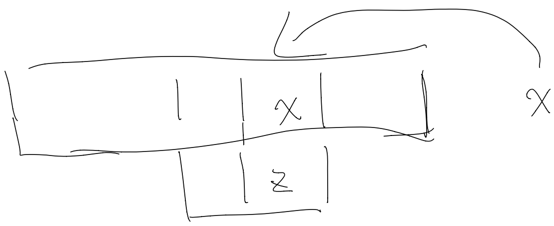




2

$\underline{T \leftarrow 2}$

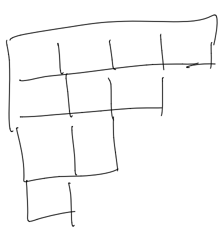
i) 

ii) 

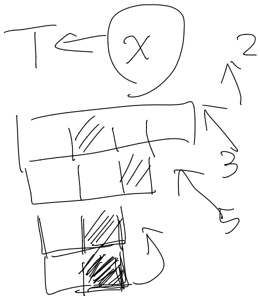
$x < y < z$

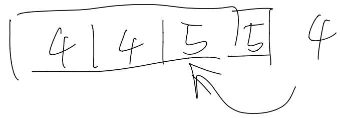
$T \leftarrow x$

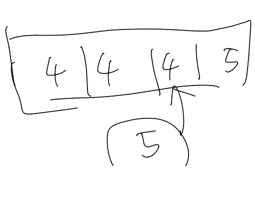
T

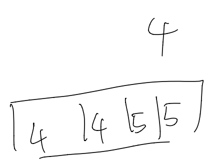
T 

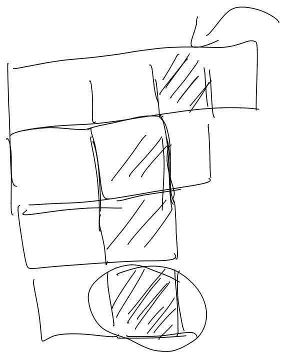
$T \leftarrow x$











x $\underline{(T \leftarrow x)}$ $\underline{T}$

bumping route new box

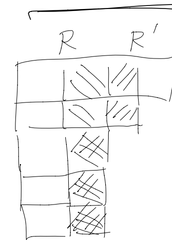
2 route R, R'

R strictly left (weakly left) of R' :

If in each row which contains a box of R'

<u>R has a box</u> which is <u>left</u> (<u>left of</u> or <u>equal to</u>)

the box in R'



Row bumping lemma.

Give 2 successive row-insertions

$$\underline{(T \leftarrow x)} \underline{\leftarrow x'}$$

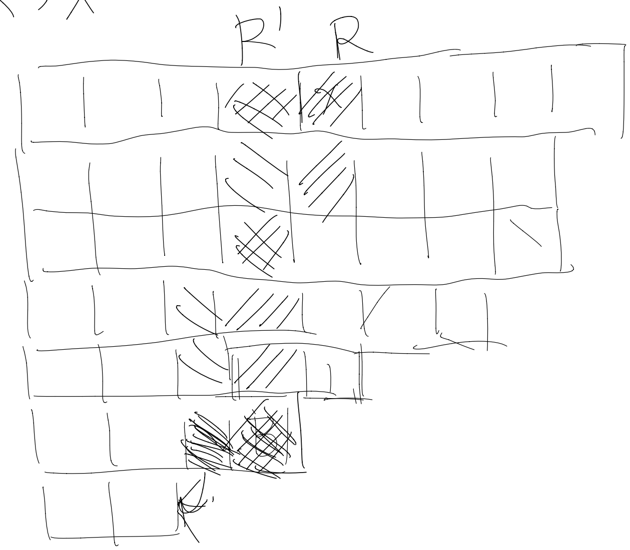
i) $x \le x'$ R strictly left of R'

B strictly left of and weakly below B'

ii) $x > x'$ R' weakly left of R

B' weakly left of and strictly below B

$x > x'$



x'

T

R x

x'

$x \le x$

$\boxed{B'}$

$x \to y$ $(x < y)$

$x' \to x = y'$

$y' \le x$

$y' \le x < y$

R strictly left of R'

B strictly left of B'

B weakly below of B'

i) R' weakly left of R

B' weakly left B

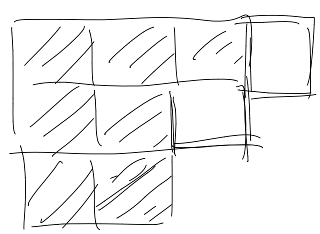
strictly below B

Skew diagram $\mu \subset \lambda$

$\mu = (\mu_1, \ldots, \mu_l)$

$\lambda = (\lambda_1, \ldots, \lambda_l)$

$\mu_i \leq \lambda_i$



λ / μ skew ~~diagram~~ $(\mu \subset \lambda)$

skew tableau

Prop: T tableau of shape λ

$$U = ((T \leftarrow x_1) \leftarrow x_2) \leftarrow \ldots \leftarrow x_p$$

U is of shape μ

① if $x_1 \leq x_2 \leq \cdots \leq x_p$ $\quad \mu / \lambda$

no 2 boxes in μ/λ in the same column

if $x_1 > x_2 > \cdots > x_p$ $\quad \mu / \lambda$

no 2 boxes in μ/λ in the same row

② conversely $\underline{U}$ of shape $\underline{\mu}$ $\lambda \subset \mu$

$\underline{\lambda}$

i) $\underline{\text{no 2 boxes in } \mu/\lambda}$ in the $\underline{\text{same column}}$

ii) no 2 boxes in μ/λ $\underline{\text{in the same row}}$

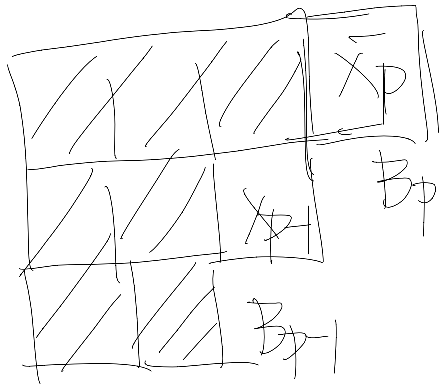
$\exists!$ tableau T of shape λ

$\exists\ x_1, \dots, x_p$ (i) $\underline{x_1 \le x_2} \le \cdots \le x_p$

(ii) $\underbrace{x_1 > x_2 > \dots > x_p}$

s.t $U = ((T \leftarrow x_1) \leftarrow x_2 \dots \leftarrow x_p$

i)



x_p

x_{p-1}

$\vdots$

x

$x_1 \le x_2 \le \dots \le x_{p-1} \le x_p$

(ii) $x_{p-1} > x_p$ $\quad B_p$

$x > x'$

$|$

B_p weakly left of B_{p-1}

T. U

1	2	3	4
2	3	4	
3	5		

u = (35 234 1234)

$$\underline{T \cdot u} = ((T \leftarrow 3) \leftarrow 5) \leftarrow \cdots \leftarrow 3 \leftarrow 4$$

Claim: The product operation makes the set of tableaux into an associative monoid.

The empty tableau is the unit element

$$\{T\} \qquad \phi \cdot T = T \cdot \phi$$

$$(T_1 \cdot T_2) \cdot T_3 = T_1 \cdot (T_2 \cdot T_3) = T$$

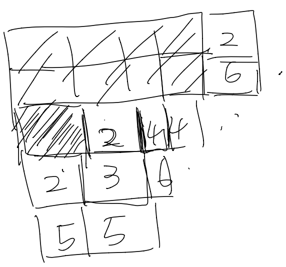
Sliding

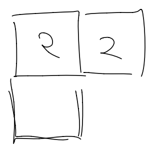
Skew diagram

inside corner

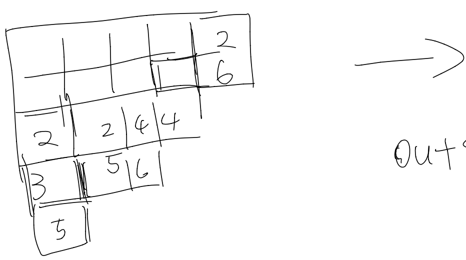
outside corner

μ / λ

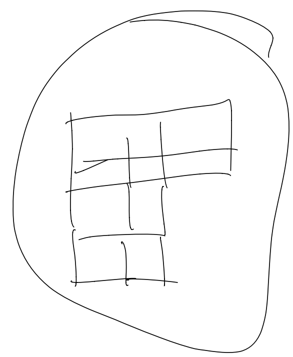




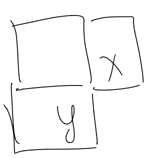
Sliding operation



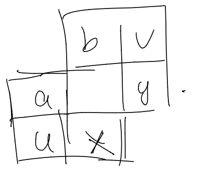
outside corner



Young diagram



Young tableaux



$b \le v < y$

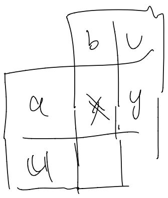
$a \le y$

$a < u \le x$

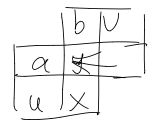
$x \le y$

$b < X$

$X > y$



$a < x \le y$



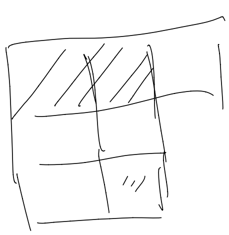
$a \le y$

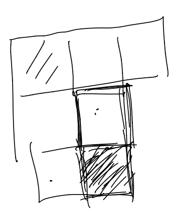
$u \le x$

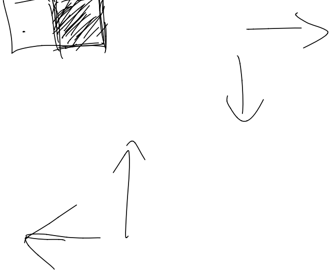
$y \cdot v$

$v < y$

skew diagram

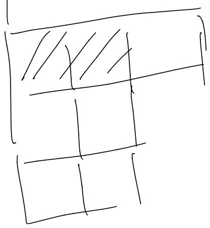






reverse slide

T



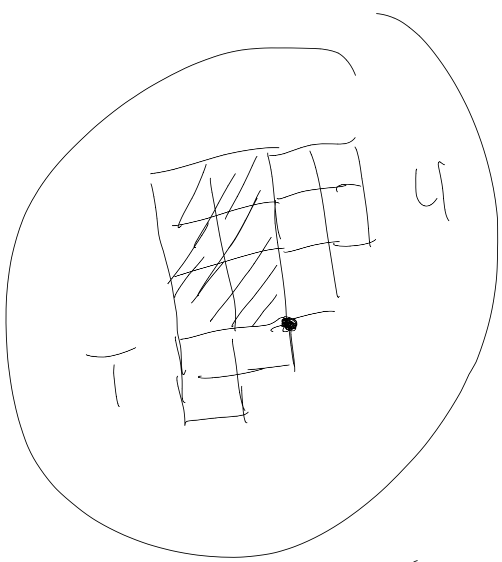
Skew tableau

$\downarrow$ slide

Young tableau = Rectification of T

Rect(T)

$$T \cdot U = \mathrm{Rect}(T * U)$$



$T * U$

skew diagram

$$T \cdot_1 U = (T \leftarrow x_1) \leftarrow \cdots \leftarrow x_p$$

$$T \cdot_2 U = \mathrm{Rect}(T * U)$$

$$\frac{\text{word}}{\text{of Young Tableaux}}$$

$$\frac{\text{free group}}{w_1 \cdot w_2} = \frac{\coprod}{\cancel{aa}}$$

$$\frac{\coprod}{\cancel{a_1 a_2 \cdot a_3} \cdot a_3 a_2 a_1} = \phi$$

$$\begin{array}{|c|c|c|}
\hline
1 & 2 & 3 \\
\hline
3 & 4 & \\
\cline{1-2}
6 & 7 & \\
\cline{1-2}
\end{array}$$

$$6\,7 \mid 3\,4 \mid 1\,2\,3$$

$$\text{Y.T.} \rightleftarrows \text{word}$$

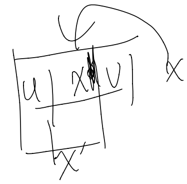
$$\begin{array}{|c|c|c|}
\cline{1-1}
x & \multicolumn{2}{c}{} \\
\hline
y & \cdots & v \\
\hline
\end{array}$$

$$\underline{x} < y \leq \underline{v}$$

$$y \cdots \underline{v \cdot x} \cdots$$

row-insert

slide



$x' \quad u \cdot x \cdot v$

$T \leftarrow x$

$u_0 \qquad x$

$u \cdot x' \cdot v \cdot x \rightsquigarrow x' \cdot u \cdot x \cdot v$

$u\, x'\, v_1 \dots v_{q-1}\, v_q\, x \xmapsto{(x < v_{q-1} \le v_q)} u\, x'\, v_1 \dots v_{q-2} v_{q-1}\, x\, v_q$

$\xmapsto{(x < v_{q-2} \le v_{q-1})} u\, x'\, v_1 \cdots v_{q-3}\, x\, v_{q-2} v_{q-1} v_q$

$\vdots \xmapsto{x < v_2 \le v_3} u\, x' v_1\, x\, v_2 \dots v_q$

$\xmapsto{x < x' \le v_1} u\, x'\, x\, v_1 v_2 \dots v_q$

$u\, x'\, x\, v$

$$u_1 \ldots u_{p-1} u_p \underline{x' x} \, v$$

$$\xrightarrow{u_p \le x < x'} u_1 \cdots u_{p-2} \underline{u_{p-1} x' u_p} x v$$

$$\xmapsto{u_{p-1} \le u_p < x'} u_1 \cdots u_{p-3} u_{p-2} x' u_{p-1} u_p x v$$

$$\vdots$$

$$\xmapsto{u_1 \le u_2 < x'} \underline{u_1 x' u_2} \cdots u_p x v$$

$$\xmapsto{u_1 \le u_2 < x'} x' u_1 u_2 \ldots u_p x v$$

$$\parallel$$

$$x' u x v$$

$\cancel{0}$

$y z x \longmapsto y x z \quad (x < y \le z) \qquad K'$

$x z y \longmapsto z x y \quad x \le y < z \qquad K''$

Knuth $\qquad ((K')^{-1} \quad (K'')^{-1}$

Knuth equivalent.

$$\boxed{y \mid z} \cdot \boxed{x} \xmapsto{x < y \le z} \begin{array}{|c|c|}\hline x & z \\ \hline y \\ \cline{1-1}\end{array}$$

row insert

K'

$$w(T) \cdot x \qquad\qquad w(T \leftarrow x)$$

$$\boxed{x \mid z} \cdot \boxed{y} \xmapsto{x \le y < z} \begin{array}{|c|c|}\hline x & y \\ \hline z \\ \cline{1-1}\end{array}$$

row insert

K''

elementary Knuth transformation

$$w \xrightarrow[(K')^{-1} \quad (K'')^{-1}]{(K') \, . \, (K'')} w'$$

Knuth equivalent $\underline{w \equiv w'}$

Prop: $T \leftarrow x$

$$\underline{w(T \leftarrow x) \equiv w(T) \cdot x}$$

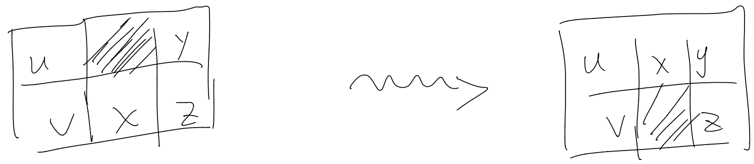
Cor $\quad T \circ U$

$$w(T \circ U) \equiv w(T) \circ w(U)$$

$$\begin{aligned}
w(T \circ U) &= w\left(\left(\left(T \leftarrow x_1\right) \leftarrow x_2 \leftarrow \dots \leftarrow x_n\right)\right) \\
&\equiv w(T \leftarrow x_1 \leftarrow \dots \leftarrow x_{n-1}) \circ x_n \\
&\equiv w(T) \circ \underline{x_1 \circ x_2 \dots x_{n-1} \circ x_n} \\
&= w(T) \circ w(U)
\end{aligned}$$

row insert preserve knuth equivalence classes of word

slide operation



K' K'' $u < v \le x \le y < z$

$vxzuy \equiv vzuxy$

$u<y<z$ $vxuzy$ $(K'')^{-1}$

$u<v\le x$ $vuxzy$ $(K'')^{-1}$

$x\le y<z$ $vuzxy$ $(K')^{-1}$

$u<x<z$ $vzuxy$ $(K')^{-1}$