Young diagram

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 4 & 5 \\ \cline{1-2}\end{array}$$

$(\lambda_1, \lambda_2, \ldots, \lambda_n) = \lambda$

$\downarrow$ filling

Young tableau

i) weakly increasing in each row

ii) strictly increasing in each column

I) row-insertion

$T \leftarrow x$

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 4 & 5 \\ \cline{1-2}\end{array} \quad \leftarrow x$$

i) $\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 3 & x \\ \hline\end{array}$ $x \geq$ numbers in the first row

ii) $\begin{array}{|c|c|c|c|c|}\hline \; & \cdots & y & \cdots & \; \\ \hline\end{array}$ x

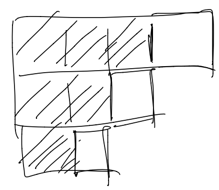
$$\Downarrow$$

$\begin{array}{|c|c|c|c|c|}\hline \; & \cdots & x & \cdots & \; \\ \hline\end{array}$ y

II sliding   skew-tableau

$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$

$\mu = (\mu_1, \mu_2, \ldots, \mu_n)$



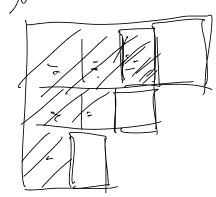
$\lambda = (4.3.2)$

$\mu = (3.2.1)$

$\mu \subset \lambda$ $\mu_i \leq \lambda_i$

λ / μ

μ



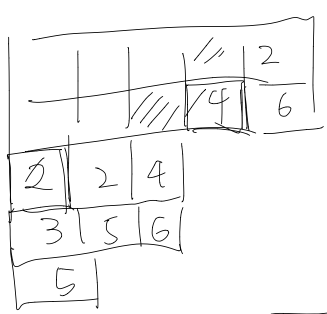
$\lambda/\mu \longrightarrow \lambda$

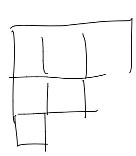
μ

λ

inside corner — μ

outside corner — μ





T · U

$x_1 x_2 \ldots x_n$

$((T \leftarrow x_1) \leftarrow x_2) \leftarrow \ldots \leftarrow x_n$

Claim 1: The product operation makes the set of tableaux into an associative monoid, The empty tableau is a unit in the moniod.

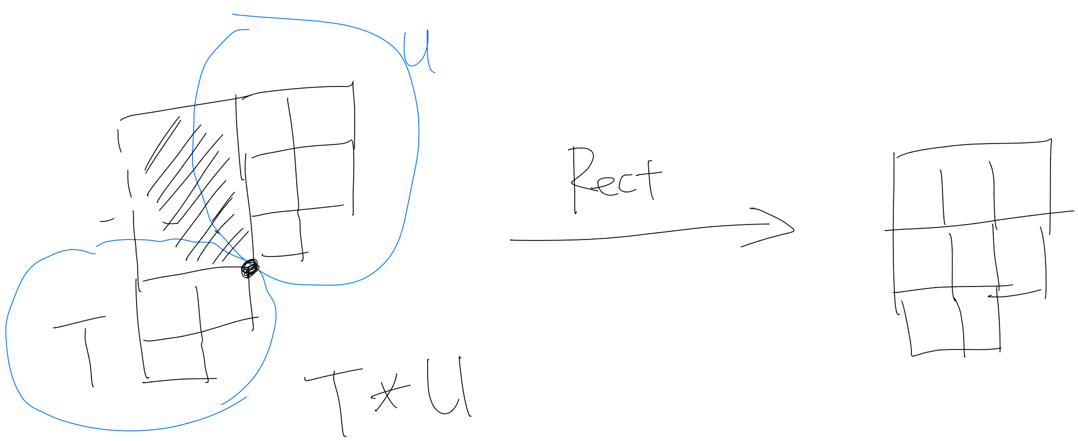
$\phi \cdot T = T \cdot \phi = T$

Claim 2: starting with a given skew tableau, all choices of inside corners lead to the same tableau.

Claim 3: These 2 products coincide.

Claim 2 + Claim 3 $\Rightarrow$ claim 1

$T \cdot_2 U = \mathrm{Rect}(T * U)$



$$\frac{\text{words}}{\text{monoid}} \quad \sim \quad \text{monoid}$$

$$\underline{T} \Longleftrightarrow \underline{\underline{\boxed{\text{word}}}}$$

Main Theorem: $\underline{T}$ is uniquely determined

by the equivalence class of

$\underline{\underline{K'}}$ $\quad (K')^{-1}$ $\quad$ its word

$\underline{K''}$ $\quad (K'')^{-1}$

Ch 3: give a proof to the Main Theorem

word associated with a tableau.

$$w \qquad\qquad w'$$
$$\| $$
$$x_1 x_2 \cdots x_m \qquad y_1 y_2 \cdots y_n$$

$$w \cdot w' = x_1 x_2 \cdots x_m y_1 y_2 \cdots y_n$$

free group $\quad w \cdot w$

$a \cdot b \cdot \cdots$

K' $\quad \underline{yzx} \longmapsto yxz \quad (x < y \le z)$

K'' $\quad xzy \longmapsto zxy \quad (x \le y < z)$

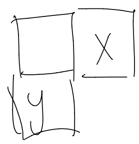
$\equiv$ Knuth equivalence

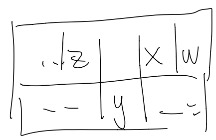
$$w(T \underset{1}{\cdot} U) \equiv w(T) \cdot w(U)$$

$$w(T \leftarrow x) \equiv w(T) \cdot x$$

I) row - insertion preserve the equivalence class of words

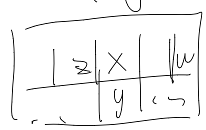
II) sliding preserve the equivalence class of words.



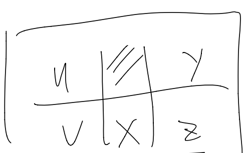


$\longrightarrow$ $z\,x\,w$

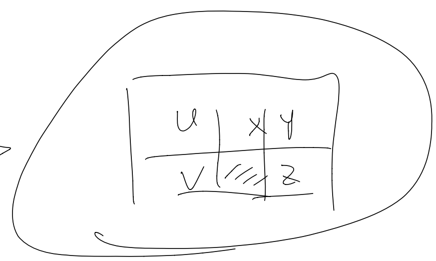
$x<y$



$\longrightarrow$ $z\,x\,w$



$\rightsquigarrow$



$u<v\le x\le y<z$

$vxzuy \xmapsto{(u<y<z)} vxuzy$ $\quad (K'')^{-1}$

$\xmapsto{(u<v\le x)} vuxzy$ $\quad K'$

$\xmapsto{(x\le y<z)} vuzxy$ $\quad K''$

$\xmapsto{u<x<z} vzuxy$ $\quad K''$

$v\,x\,z\,u\,y \equiv v\,z\,u\,x\,y$

$$\begin{array}{|c|c|c|c|c|c|c|}\hline u_1 & \cdots & u_p & & y_1 & \cdots & y_q \\ \hline v_1 & & v_p & x & z_1 & \cdots & z_q \\ \hline\end{array} \rightsquigarrow \begin{array}{|c|c|c|c|c|c|c|}\hline u_1 & \cdots & u_p & x & y_1 & \cdots & y_q \\ \hline v_1 & & v_p & & z_1 & & z_q \\ \hline\end{array}$$

$u_i < v_i \quad y_j < z_j \qquad v_p \le x \le y_1$

$u = u_1 \cdots u_p \quad v = v_1 \cdots v_p \qquad y = y_1 \cdots y_q \quad z = z_1 \cdots z_q$

$v\,\underline{x\,z}\,u\,y \equiv v\,z\,u\,x\,y$

induction on p

① $p = 0 \quad \underline{x\,z_1 \cdots z_q\,y_1 \cdots y_q} \equiv z_1 \cdots z_q\,x\,y_1 \cdots y_q$

$\underline{x\,z_1 \cdots z_q}\; y_1 \cdots y_q$

$(x\,z_1 \cdots z_q) \leftarrow y_1 \equiv z_1\,x\,y_1\,z_2 \cdots z_q$

$(z_1\,x\,y_1\,z_2 \cdots z_q)\,y_2 \cdots y_q$

$(((y_2, \ldots, y_q$ insert

$z_1 z_2 x y_1 y_2 z_3 \cdots z_q$

$\equiv$

$\cdots$

$\equiv$

$z_1 z_2 \cdots z_q x y_1 \cdots y_q = zxy$

$p \geq 1 \quad vxzuy \equiv vzuxy \quad (*)$

assume $(*)$ is true for smaller p

$u' = u_2 \cdots u_p$

$v' = v_2 \cdots v_p$

$$vxzuy = \underline{v_1 v' xz u_1 u' y}$$

$$v_1 v' xz \leftarrow u_1 \equiv v_1 u_1 v' xz$$

$$vxzuy \equiv v_1 u_1 v' \underline{xzu'y}$$

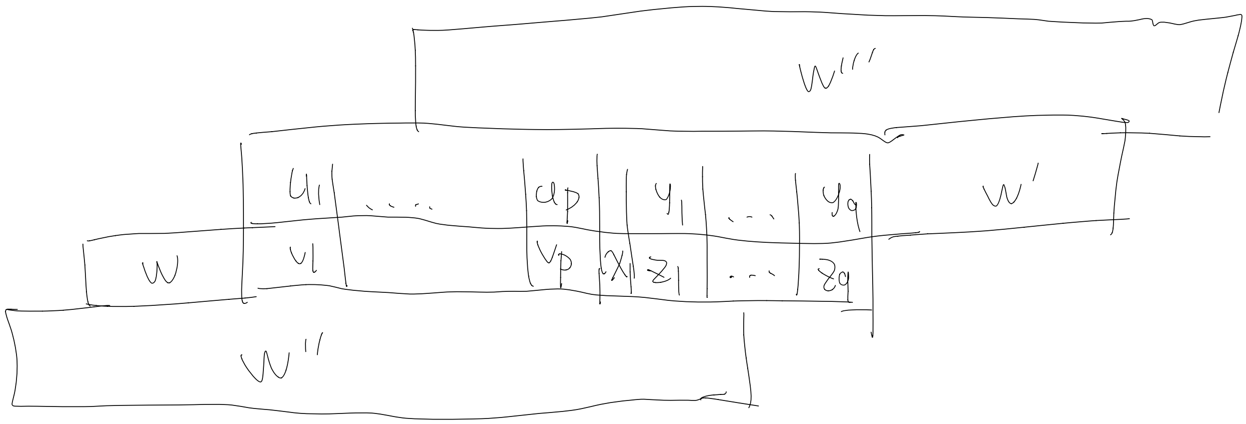
$$vxzuy \equiv vzuxy$$

$$v'xzu'y \equiv v'zu'xy$$

$$\underline{v_1 u_1 v' z} u' xy$$

$$v_1 v' z \leftarrow u_1 \equiv \underline{v_1 u_1 v' z}$$

$$vzuxy = v_1 v' z u_1 u' xy$$



$w''\, w\, vxzuyw'\, w'''$

$\equiv$

$w''\, w\, vzuxyw'\, w'''$

Prop 2: If one (skew) tableau can be obtained from another (skew) tableau by a sequence of slides, then their words are knuth equivalent

Theorem: Every word is knuth equivalent to the word of a unique tableau.

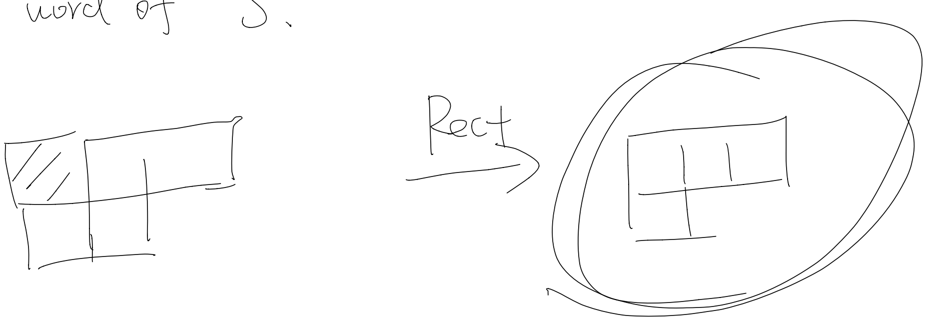
$w = x_1 x_2 \cdots x_r$

$$\underline{T} = \Big(((x_1 \leftarrow x_2) \leftarrow \cdots \leftarrow x_r \Big)$$

$w(T) \equiv w$

canonical procedure for constructing a tableau whose word is equivalent to a given word.

Cor 1: The rectification of a skew tableau S is the unique tableau whose word is equivalent to the word of S.



$\underline{T \cdot_3 U} =$ the unique tableau whose word is equivalent to $w(T) \cdot w(U)$

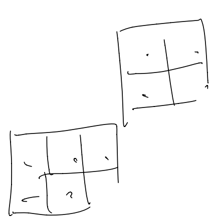
Cor 2: The three constructions of the product of two tableaux agree.

$$T \cdot_1 U = T \cdot_3 U = T \cdot_2 U$$

$$w(T \cdot_1 U) \equiv w(T) \cdot w(U)$$

$$\Rightarrow \quad T \cdot_1 U = T \cdot_3 U$$

$$T \cdot_2 U = \underline{\mathrm{Rect}} (T * U)$$



$T * U$

$$\underline{w(T * U)} = w(T) \cdot w(U)$$

$\qquad\qquad\equiv$ by sliding operation

$$w(\mathrm{Rect}(T * U))$$

$$\|$$

$$w(T \cdot U)$$

$$\Rightarrow T \cdot_2 U = T \cdot_3 U$$

plactic monoid

$M = M_m = \{$ equivalence classes of words on the alphabet $[m] = \{1, 2, \dots, m\}$ $\}$

(associate monoid)

$w \cdot w$

$w \equiv w'$ $\qquad$ $w \cdot v \equiv w' \cdot v \equiv w' \cdot v'$

$v \equiv v'$

$$M \xleftarrow[\sim]{\phi} \text{associative monoid of tableau}$$

① $T \longrightarrow w(T)$

$T \cdot U \longrightarrow w(T \cdot_1 U)$

$$\|$$

$$w(T) \cdot w(U)$$

monoid of tableaux

monoid of words

$[m] = \{1, 2, \dots, m\}$

$\{ x_{i_1} x_{i_2} \dots x_{i_k} \; ; \; x_{i_j} \in [m] \}$

$x_{i_1} x_{i_2} \dots x_{i_k} x_{i_k} x_{i_{k-1}} \dots x_{i_2} x_{i_1} = \phi$

$x_{i_1} x_{i_2} \dots x_{i_k} x_{i_1} \dots x_{i_k}$

$x_{i_1} x_{i_1} = \phi$

$\phi(T \cdot U) = \phi(T) \cdot \phi(U)$

Theorem says ϕ is bijective

group ring

$R_{[m]}$ (tableau ring)

$\|$

$\bigoplus \mathbb{Z}_i$ ($\mathbb{Z}_i = \mathbb{Z}$

with basis the ~~t~~ tableaux with

entries in $[m] = \{1, 2, \dots, m\}$)

$$R_{[m]} \longrightarrow \mathbb{Z}[x_1, x_2, \dots, x_m]$$

$$T \longmapsto x^T$$

$$\| \\ \prod (x_i)^{\text{the number of } i \text{ occurs in } T}$$

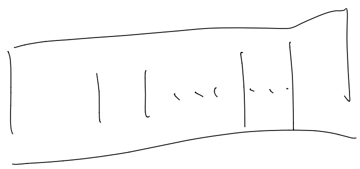
Schur polynomial — tableau ring

$S_\lambda = S_\lambda[m] \in R[m]$

$\|$

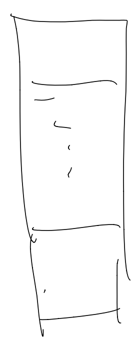
$\sum$ all tableaux T of shape λ with entries in $[m]$

(p)



p boxes

(1^p)



p boxes.

$S_{(p)}$ = pth complete symmetric polynomial

$\|$ $h_p(x_1, \dots, x_m)$

$S_{(1^p)}$ = pth elementary symmetric polynomial

$\|$

$e_p(x_1, \dots, x_m)$

$\boxed{1 \cdots 1}$ (p)

$\sum \prod (X_i)^{\text{number of times } i \text{ occur in } (p)}$

$= S_{(p)}$

T

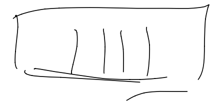
$$\sum_{k_1+\cdots+k_m = p} \prod X_i^{k_i}$$

$$\sum_{1 \le i_1 < i_2 < \cdots < i_p \le m} X_{i_1} X_{i_2} \cdots X_{i_p}$$

$m = 3$ $p = 2$

$X_1X_2 + X_1X_3 + X_2X_3$

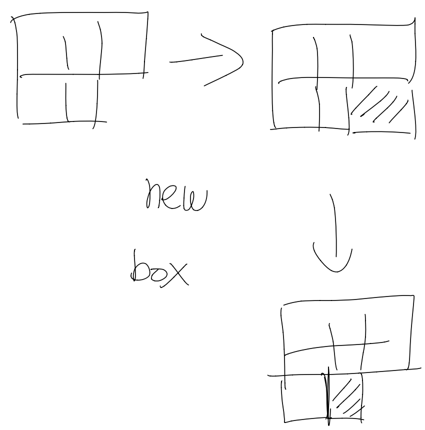
Pieri formula

$R[m]$ 

$$S_\lambda \cdot S_{(p)} = \sum_\mu S_\mu$$

μ are obtain from λ by adding p boxes, with no 2 in the same column.

Row bumping lemma

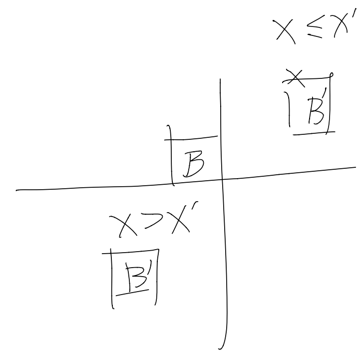


$$T \leftarrow x \leftarrow x'$$

i) $x \le x'$

B strictly left of and weakly below B'

ii) $x > x'$ B' weakly left of strictly below B



$$S_\lambda \cdot S_{(p)} = \sum S_\mu$$

μ obtained from λ by adding p boxes,

with no 2 in the same row.

$$x_p \, x_{p-1} \cdots x_1$$

$$(1^p) \qquad \begin{array}{|c|}\hline x_1 \\ \hline \vdots \\ \hline x_p \\ \hline \end{array}$$

$$S_\lambda \cdot S_{(1^p)}$$

$$S_\lambda \leftarrow \underline{x_p \leftarrow x_{p-1}} \leftarrow \cdots \leftarrow x_1$$

$$R[m] \longrightarrow \mathbb{Z}[x_1, \ldots, x_m]$$

$$s_\lambda(x_1, \ldots, x_m) \cdot h_p(x_1, \ldots, x_m) = \sum_\mu s_\mu(x_1, \ldots, x_m)$$

μ obtained no 2 in the same column

$$s_\lambda(x_1, \ldots, x_m) \cdot e_p(x_1, \ldots, x_m) = \sum_\mu s_\mu(x_1 \ldots x_m)$$

no 2 in the same row

A tableau T has content $\mu = (\mu_1, \ldots, \mu_l)$

$\lambda.\ \mu$

if its entries consist of μ_1 copies of 1

μ_2 copies of 2

$\vdots$

μ_l copies of l

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 4 & 4 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \qquad \mu = (2, 2, 3, 4)$$

Kostka number $K_{\lambda\mu}$

$=$ # tableaux of shape λ with content μ

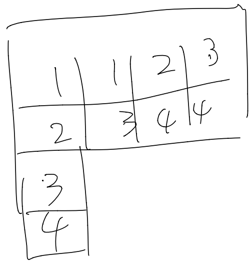
$=$ # sequences of partitions $\lambda^{(1)} \subset \lambda^{(2)} \subset \cdots \subset \lambda^{(l)} = \lambda$

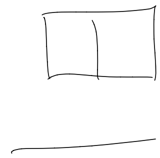
skew diagram $\lambda^{(i)} / \lambda^{(i-1)}$ has μ_i boxes.

with no 2 in the same column.

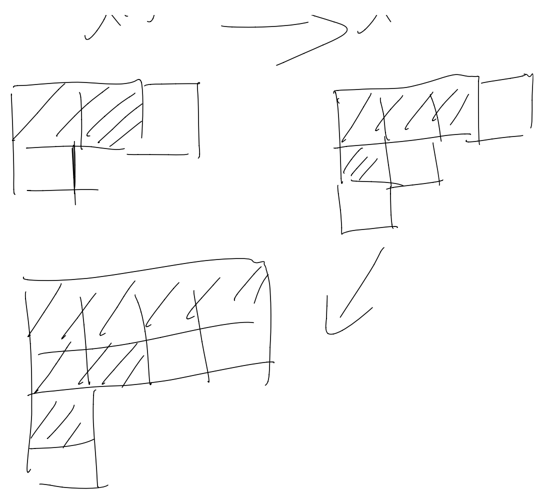
$\mu = (2, 2, 3, 3)$

$\lambda^{(1)}$ $\qquad \lambda^{(2)}$ $\qquad \lambda^{(3)}$





$\lambda^{(4)} =$



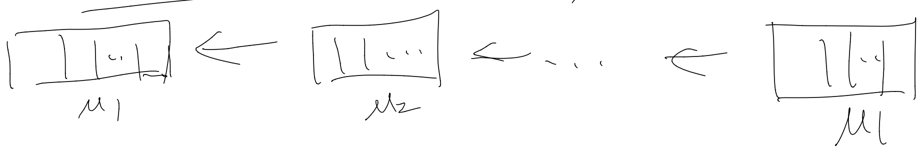
$K_{\lambda\mu}$

(6) $s_\lambda(x_1,\dots,x_m)\cdot h_p(x_1,\dots,x_m)=\sum_\mu s_\mu(x_1,\dots,x_m)$

μ obtained by adding p boxes to λ

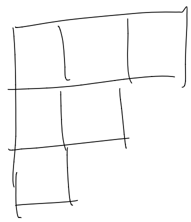
with no 2 in the same column.

(8) $h_{\mu_1}\cdot h_{\mu_2}\cdot \dots \cdot h_{\mu_l} = \sum_\lambda K_{\lambda\mu}\, s_\lambda$



$\lambda \qquad 1,\dots,l \qquad$ λ has content $\mu=(\mu_1,\dots,\mu_l)$

$\qquad \mu_1 \quad \mu_l \qquad \mu$

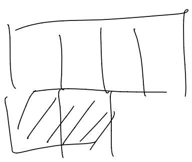


λ

$\underset{\text{boxes}}{\mu_1} \leftarrow \underset{\text{boxes}}{\mu_2} \leftarrow \cdots \leftarrow \mu_l$ box

$\Downarrow$

T of shape λ

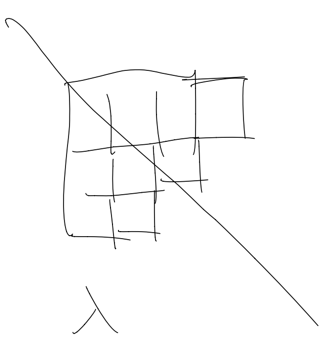


$K_{s\lambda}\ S_\lambda$

$K_{s\lambda}=0 \quad \lambda$

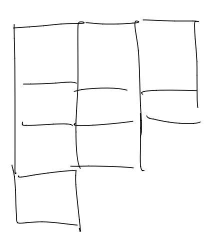
$$e_{\mu_1}\cdot e_{\mu_2}\cdots e_{\mu_l} = \sum_{\lambda} K_{\tilde{\lambda}\mu}\, s_\lambda = \sum K_{\lambda\mu}\, s_{\tilde{\lambda}}$$

$\tilde{\lambda}$ is the conjugate to a partition λ



λ

$(4,3,2)$



$(3,3,2,1)$

$$s_\lambda \cdot s_{(p)} = \sum s_\mu$$

μ obtained from λ by adding p boxes,

with no 2 in the same row.

$$s_\lambda \cdot e_p = \sum_\mu s_\mu$$

ordering on the tableaux.

$x, y \in A$

partial ordering "$\leq$" A

$x \leq y$

i) $x \leq x$ (reflective)

$y \leq x$

ii) $x \leq y$ $y \leq x$ $\Rightarrow x = y$ (anti-symmetric)

iii) $x \leq y$ $y \leq z \Rightarrow x \leq z$ (transtive) total ordering

partial ordering $\dagger$

$\forall x, y \in A \quad x \leq y$ or $y \leq x$

$\mu \subset \lambda$ inclusion $\mu_i \leq \lambda_i$

$\mu \trianglelefteq \lambda$ dominance ordering

$\forall i \quad \mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i$

$\mu \leq \lambda$ lexicographic ordering

for the first i s.t. $\mu_i \neq \lambda_i$

$\mu_i < \lambda_i$

$$\mu \subset \lambda \underset{\not\Leftarrow}{\Rightarrow} \mu \trianglelefteq \lambda \underset{\not\Leftarrow}{\Rightarrow} \mu \leq \lambda$$

$\lhd$ $\top$

Exer 2): for partitions λ, μ of the same integer.

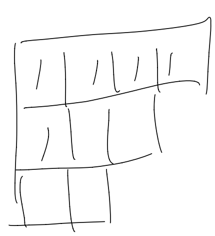
$K_{\lambda\mu} \neq 0 \iff \mu \trianglelefteq \lambda$

$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$

$\mu = (\mu_1, \mu_2, \ldots, \mu_l)$

$\mu \trianglelefteq \lambda$

$\mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i \quad \forall i = 1, \ldots, l$



4
3
2

μ

$\mu_1 \leq \lambda_1$

$\mu_1 + \mu_2 \leq \lambda_1 + \lambda_2$

$\vdots$

$\mu_1 + \cdots + \mu_l \leq \lambda_1 + \cdots + \lambda_l$

for some j

$\mu_1 + \cdots + \mu_j > \lambda_1 + \cdots + \lambda_l$

assume $j = 1$ $\quad \mu_1 > \lambda_1 = 4$

$\mu_1 \geq 5$

T of shape λ content μ

μ_1 copies of 1

μ_2 copies of 2

$\vdots$

μ_l copies of l

$\mu_1+\cdots+\mu_{j-1} \le \lambda_1+\cdots+\lambda_{j-1}$

$\boxed{\mu_1+\cdots+\mu_j > \lambda_1+\cdots+\lambda_j}$

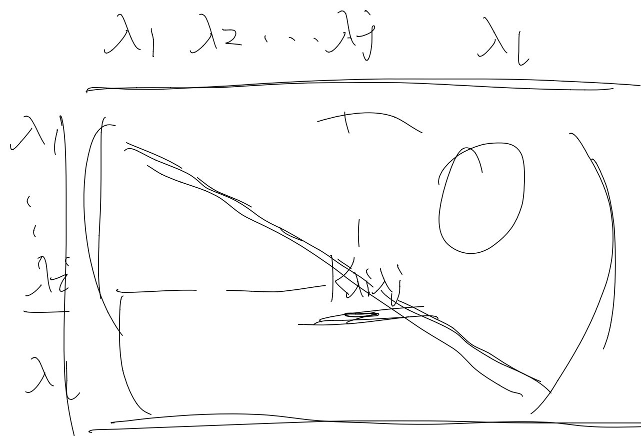
the number j will occur in the row $\geq j+1$

j

Schur polynomials are symmetric polynomials

partions with dominance ordering

$(K_{\lambda\mu} \neq 0 \iff \mu \trianglelefteq \lambda)$



$= K$ non-singular

$K_{\lambda_i\lambda_j}$

$\lambda_i < \lambda_j$

$K_{ij} = K_{\lambda_i\lambda_j}$

$\lambda = \mu \quad K_{\lambda\mu} = K_{\lambda\lambda} =$

(8) $\underline{h_{\mu_1}\cdot h_{\mu_2}\cdot \ldots \cdot h_{\mu_l}} = \sum_{\lambda} K_{\lambda\mu}\, s_\lambda$

$$\| \quad h_\mu$$

$(h_{\mu_1},\ldots,h_{\mu_l}) = (s_{\lambda_1},\ldots,s_{\lambda_l})\, K$

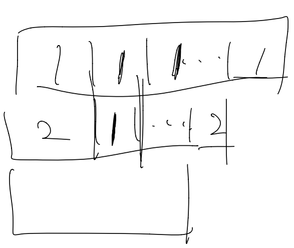
$\lambda=\mu \qquad K_{\lambda\mu}=K_{\lambda\lambda}=1$

$\mu \trianglelefteq \lambda$

$\lambda=(\lambda_1,\ldots,\lambda_l) \qquad \mu_1+\cdots+\mu_i=\lambda_1+\cdots+\lambda_i$

$\|$

$\mu=(\mu_1,\ldots,\mu_l) \qquad \mu_j=\lambda_j$

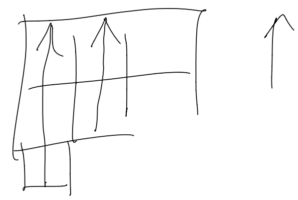


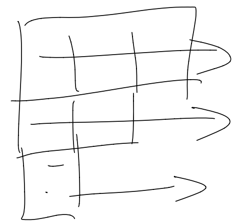
$\lambda_1=\mu_1$

$\lambda_2=\mu_2$

$(s_{\lambda_1},\ldots,s_{\lambda_l}) = (h_{\mu_1},\ldots,h_{\mu_l})\, K^{-1}$

Schur polys are symmetric

Column words 

row words 

$$w_{col}(T) \equiv w_{row}(T)$$

Knuth equivalent.

Theorem: Every word is Knuth equivalent to the word of a unique tableau.

Increasing sequence in a word

$$w = x_1 x_2 \ldots x_r$$

$$i_1 < i_2 < \cdots < i_l$$

$$x_{i_1} \leq x_{i_2} \leq \cdots \leq x_{i_l}$$ Increasing sequence

$L(w,k)$ = the largest number which is the sum of the length of k disjoint increasing sequences in w.

$w = 1\,3\,4\,2\,3\,4\ \ 1\,2\,2\ \ 3\,3\,2$

$L(w,1) = 6$

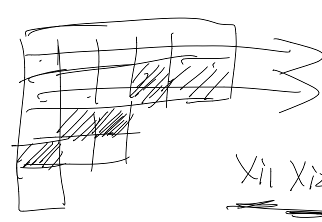
$1\,3\,4\,2\,3\,4\ \ 1\,2\,2\ \ 3\,3\,2$

$L(w,2) = 9$

$L(w,3) = 12$ $\quad 1\,3\,4\ \ 2\,3\,4\ \ 1\,2\,2\ \ 3\,3\,2$

$L(w,k)$ any increasing sequence taken from w must consist of numbers that are taken from the tableau in order from left to right, never going down in the tableau.

$1\,3\,4 \mid 2\,3\,4 \mid 1\,2\,2\,3\,3 \mid 2$



$x_{i1}\ x_{i2}\ \cdots\ x_{ic}$

$L(w,1) = \#$ column

$x_{i1}\, x_{i2}$ $\quad$ $\begin{array}{c} x_{i1} \\ \boxed{x_{i2}} \\ \boxed{x_{i1}} \end{array}$

$x_{i2} < x_{i1}$

$L(w,k) = \#$ boxes in the first k rows

lemma 1: $\quad w \longrightarrow T \longrightarrow \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$

$\forall k \in \mathbb{Z}_{>0} \quad L(w,k) = \lambda_1 + \lambda_2 + \cdots + \lambda_k$

lemma 2: $\quad w \equiv w' \qquad L(w,k)$

$L(w,k) = L(w',k)$

K' $\quad$ i) $u \cdot yxz \cdot v \equiv u \cdot yzx \cdot v \qquad (x < y \leq z)$

K'' $\quad$ ii) $u \cdot xzy \cdot v \equiv u \cdot zxy \cdot v \qquad (x \leq y < z)$

for K' $\quad w' \quad \tilde{u} \quad \tilde{v}$

$L(w,k) \geq L(w',k)$

$L(w',k) \geq L(w,k)$

lemma 3: $w \equiv w'$

w_0, w_0' are (resp.) obtained by

removing the p largest,

q smallest numbers from w, w'

$\Rightarrow$ $w_0 \equiv w_0'$

pf:

K' i) $u\,yxz \cdot v \equiv u \cdot yzx \cdot v$ $(x < y \leq z)$

K'' ii) $u \cdot xzy \cdot v \equiv u \cdot zxy \cdot v$ $(x \leq y < z)$

w w'

if remove the biggest number in x, y, z

K' $w_0 = u\,yxv$

$\parallel$

$w_0' = u\,yxv$

K'' $\quad w_0 = uxyv$

$\qquad\quad w_0' = uxyv$

Theorem:

T is determined by its word

T has shape λ

has filling

$w \equiv w(T)$

lemma 2:
$w \equiv w', \quad L(w,k) = L(w',k)$

lemma 1: T of shape $\lambda = (\lambda_1, \ldots, \lambda_l)$

$L(w,k) = \lambda_1 + \cdots + \lambda_k$

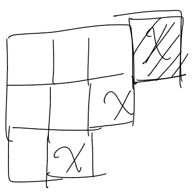
$\lambda_k = L(w,k) - L(w,k-1)$

assume $\underline{x}$ largest number in T

w_0 the word left by removing the right-most occurance of x from w.

T_0. obtained by removing x from T from the position farthest to the right.

$w(T_0) = w(T)_0$



lemma 3 $\Rightarrow$ $\underline{w_0} \equiv w(T)_0 = \underline{w(T_0)}$

induction on the length of the word

$w(T_0) \equiv w_0$

T_0 is the unique tableau whose word is equivalent to w_0

the shape of T_0 and T are known

$\Rightarrow T$