The Robinson – Schensted – Knuth correspondence

R – S – K correspondence

w $\quad \underline{P(w)}$ = unique tableau whose word is Knuth equivalent to w

$$w = x_1 x_2 \cdots x_r$$

$$P(w) = \left(\left(\boxed{x_1} \leftarrow x_2\right) \leftarrow x_3\right) \leftarrow \cdots \leftarrow x_r$$

$P(w)$, $\underline{Q(w)}$: recording tableau or insertion tableau.

$P(w)$

$$\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & \\ \cline{1-2} \\ \cline{1-1} \end{array}$$

$Q(w)$

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array}$$

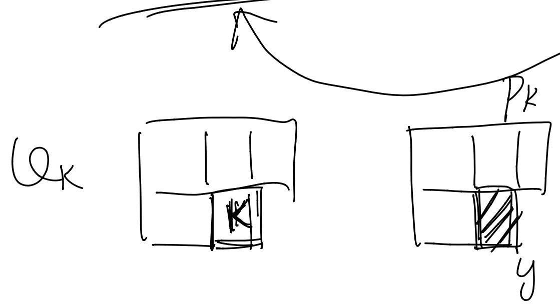
$$w \longrightarrow (P(w), Q(w))$$
$$\longleftarrow$$

(P_k, Q_k)

$$\underline{w} \to (P_1, Q_1) \to (P_2, Q_2) \to (P_3, Q_3)$$

$$\to \cdots \to (P_s, Q_s) = \underline{(P, Q)}$$

$(P_k, Q_k) \longrightarrow (P_{k-1}, Q_{k-1})$



$(P_{k-1} \leftarrow x_k) \longrightarrow P_k$

row-bump

$w = x_1 \cdots x_{s-1} x_s$

Robinson-Schensted correspondence.

words of length r using the letters from $[n] = \{1, 2, \dots, n\}$ $\xleftrightarrow{1=1}$ (ordered) pairs (P, Q) of tableaux of the same shape with r boxes, with the entries of P taken from $[n]$, Q standard

Robinson correspondence

$r = n$ $\quad \begin{pmatrix} 1 & 2 & \cdots & n \\ w_1 & w_2 & & w_n \end{pmatrix} \in S_n$

w_i different from each other

$w_i \in [n]$

permutations $\Longleftrightarrow$ pairs of standard tableaux.

$\Downarrow$

Robinson– Schensted– Knuth correspondence

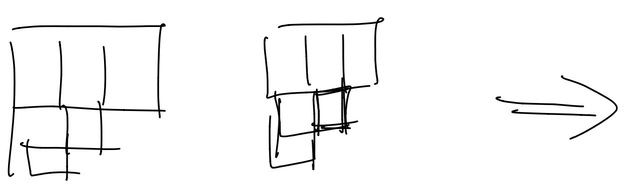
P has entries from $[n] = \{1, 2, \dots, n\}$

Q has entries from $[m] = \{1, 2, \dots, m\}$

$\underline{(P, Q)} = \underline{(P_r, Q_r)}, (P_{r-1}, Q_{r-1}), \dots, \underline{(P_1, Q_1)}$

using (P, Q) to find the word w.

$\underline{(P_k, Q_k)} \Longrightarrow \underline{(P_{k-1}, Q_{k-1})}$



find the largest entry from $\underline{Q_k}$, denote by $\underline{y_k}$.

find the enty of P_k with the same place of y_k.

assume z_k

row-bumping   reverse row-bump $z_k \Rightarrow P_k$

2-rowed x_k array

$$\left(\begin{array}{cccc} u_1 & u_2 & \cdots & u_r \\ \hline v_1 & v_2 & \cdots & v_r \end{array}\right)$$

the k-th step. u_k are deleted from Q_k

v_k are deleted from P_k

if Q is standard

the array is $\begin{pmatrix} 1 & 2 & \cdots & r \\ v_1 & v_2 & \cdots & v_r \end{pmatrix}$

$\Rightarrow$ $w = v_1 v_2 \cdots v_r$

for 2-rowed array $\left(\begin{array}{c} u_1, u_2, \ldots, u_r \\ \hline v_1, v_2, \ldots, v_r \end{array}\right)$

$u_i = i$

$\{v_j\}$ are different $\{v_j\} = [r]$

this kind of array is permutation.

$$\begin{pmatrix} u_1, u_2, \dots, u_r \\ v_1, v_2, \dots, v_r \end{pmatrix} \longleftarrow (P, Q)$$

i) $u_1 \le u_2 \le \cdots \le u_r$

ii) if $u_{k-1} = u_k$ then $v_{k-1} \le v_k$.

ii) is from Row bumping lemma

$\because$ if $u_{k-1} = u_k$

$$(P_k, Q_k) \longrightarrow (P_{k-1}, Q_{k-1}) \longrightarrow (P_{k-2}, Q_{k-2})$$

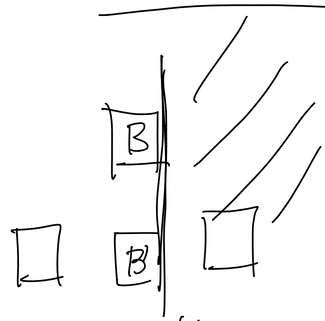
B' u_k. B u_{k-1}

B' is strictly right of B

$P_{k-2} \leftarrow v_{k-1} = P_{k-1}$ B

$P_{k-1} \leftarrow v_k = P_k$ B'

$v_{k-1} > v_k \Rightarrow B'$ weakly left of B



2-rowed array $w = \begin{pmatrix} u_1 \cdots u_r \\ v_1 \cdots v_r \end{pmatrix}$ is in lexicographic order

if i.) $u_1 \le u_2 \le \cdots \le u_r$

ii) if $u_{k-1} = u_k$ then $v_{k-1} \le v_k$.

w in lexicographic order $\rightleftarrows (P, Q)$

$\|$

$\begin{pmatrix} u_1 \, u_2 \cdots u_r \\ v_1 \, v_2 \cdots v_r \end{pmatrix}$

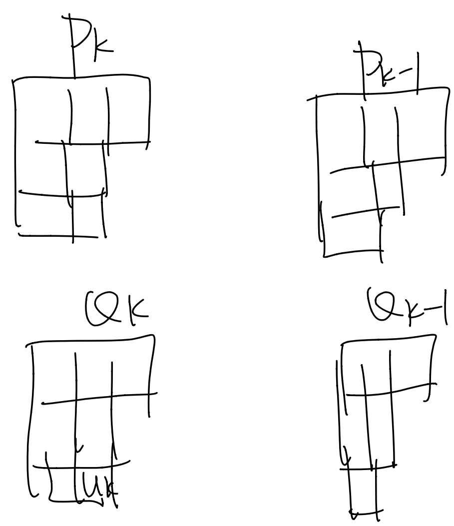
$w \to (P_1, Q_1) \to (P_2, Q_2) \to \cdots \to (P_r, Q_r) = (P, Q)$

$P_1 = \boxed{v_1}$ $\qquad Q_1 = \boxed{u_1}$

assume (P_{k-1}, Q_{k-1})

$\underline{P_k} = \underline{P_{k-1}} \leftarrow u_k$

P_k $\quad$ new box $\quad$ u_k



P is Young tableau

need check Q is Y t.

R-S-K Thm.

We have

2-rowed lexicographic arrays w

$$\Updownarrow \; 1:1$$

(ordered) pairs of tableaux (P, Q) with the same shape

Under this correspondence

i) w has r entries in each row $\Leftrightarrow$

P, Q each has r boxes.

and The entries of P are elements of the bottom row of w

the entries of Q are elements of ~ top

ii) w is a word $\Leftrightarrow$ Q is a standard tableau.

$$\begin{pmatrix} 1 & 2 & \cdots & r \\ v_1 & v_2 & \cdots & v_r \end{pmatrix}$$

iii) w is a permutation $\Leftrightarrow$ P, Q. standard tableaux.

$$\begin{pmatrix} 1 & 2 & \cdots & r \\ v_1 & v_2 & \cdots & v_r \end{pmatrix}$$

$\{v_i\} = [r] = \{1, 2, \ldots, r\}$

Symmetry Theorem:

if an array $\begin{pmatrix} u_1 & u_2 & \cdots & u_r \\ v_1 & v_2 & \cdots & v_r \end{pmatrix} \longrightarrow (P, Q)$

then $\begin{pmatrix} v_1 & v_2 & \cdots & v_r \\ u_1 & u_2 & \cdots & u_r \end{pmatrix} \longrightarrow (Q, P)$

Use the language of matrix.

$W = \begin{pmatrix} u_1 & u_2 & \cdots & u_r \\ v_1 & v_2 & \cdots & v_r \end{pmatrix}$ every 2 rowed array can be identified with a collection of pairs of elements (i, j)

$W_k = \begin{pmatrix} u_k \\ v_k \end{pmatrix}$

$W = (W_1, W_2, \ldots, W_r)$

the elements of the first row of W are entries of Q, they are from $[m]$

2nd row of W are entries

of p. ——— $[n]$

construct $m \times n$ matrix A.

A_{ij} = # times that $\binom{i}{j}$ occurs in the array w.

$$\begin{pmatrix} 1\,1\,1\,2\,2 & 3\,3\,3\,3 \\ 1\,2\,2\,1\,2 & 1\,1\,1\,2 \end{pmatrix} \begin{matrix} [3] \\ [2] \end{matrix}$$ in lexicographic order

$m=3$ $n=2$

3×2

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 3 & 1 \end{pmatrix}$$

$\underline{A_{ij}}$ $\binom{i}{j}$

R-S-K.

matrices of nonnegative integer entries

$\xleftrightarrow{1:1}$ (ordered) pairs (P, Q) of tableaux of the same shape.

A. ith row sum of A

= # times that i occurs in the top row of the array

$=$ # times that i occurs in Q

j^{th} column sum of A

$=$ # times that j occurs in P

$\tau = S_n$

$\tau^2 = 1$ $\quad$ τ is involution

$$\begin{pmatrix} 1 & 2 & 3 & 4 \cdots & r \\ \tau(1) & \tau(2) & \tau(3) & \tau(k) \cdots & \tau(r) \end{pmatrix}$$ 2-rowed array

$k \to \tau(k) \to k$ $\quad$ $\begin{pmatrix} i \\ j \end{pmatrix}$ $\begin{pmatrix} j \\ i \end{pmatrix}$

$i \to j \to i$

$\Rightarrow A$ symmetric matrix

each permutation in $S_n \longleftrightarrow$ permutation matrix

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & n \\ \tau(1) & \underline{\tau(2)} & & \tau(n) \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ \tau(1) \end{pmatrix} \begin{pmatrix} 2 \\ \tau(2) \end{pmatrix} \cdots \begin{pmatrix} n \\ \tau(n) \end{pmatrix}$$

$$\begin{array}{c} \tau(1) \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ & & & & & & \\ & & & \ddots & & & 1 \end{pmatrix} \end{array}$$

involution $\longleftrightarrow$ symmetric permution matrix

$\longleftrightarrow (P, P)$

symmmetric thm

$$\boxed{\begin{pmatrix} 1 & 2 & \cdots & r \\ v_1 & \cdots & & v_r \end{pmatrix}} \longrightarrow (P, Q)$$

$$\begin{pmatrix} v_1 & \cdots & v_r \\ 1 & 2 & \cdots & r \end{pmatrix} \longrightarrow \underline{(Q, P)} = (P, Q)$$

$$\begin{pmatrix} v_1 & \cdots & v_r \\ 1 & 2 & \cdots & r \end{pmatrix}$$ 2-rowed array

may in lexi order

in lexi

may not in lexi order

$\downarrow$ order

$v_1 \to v_2 \to v_1$

$$\begin{pmatrix} 1 & 2 & \cdots & r \\ \tau(1) & \tau(2) & \cdots & \tau(r) \end{pmatrix}$$

$\tau(i) \to i \to$

$$\begin{pmatrix} 1 & 2 & \cdots & r \\ v_1 & \cdots & & v_r \end{pmatrix} \longrightarrow \begin{pmatrix} v_1 & \cdots & v_r \\ 1 & & r \end{pmatrix}$$

$i \to v_i \to i$

$\downarrow$ in lexi order

$$\begin{pmatrix} 1 & r \\ \tau(1) & \tau(r) \end{pmatrix}$$

$\tau(i) = v(i)$

involution $\longleftrightarrow$ pair of standard tableaux

(RP)

standard tableau